ON THE SPECIAL VALUES OF CERTAIN RANKIN–SELBERG L-FUNCTIONS AND APPLICATIONS TO ODD SYMMETRIC POWER L-FUNCTIONS OF MODULAR FORMS

A. RAGHURAM

Abstract. We prove an algebraicity result for the central critical value of certain Rankin-Selberg L-functions for $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$. This is a generalization and refinement of the results of Harder [15], Kazhdan, Mazur and Schmidt [23], Mahnkopf [29], and Kasten and Schmidt [22]. As an application of this result, we prove algebraicity results for certain critical values of the fifth and the seventh symmetric power L-functions attached to a holomorphic cusp form. Assuming Langlands' functoriality one can prove similar algebraicity results for the special values of any odd symmetric power L-function. We also prove a conjecture of Blasius and Panchishkin on twisted L-values in some cases. We comment on the compatibility of our results with Deligne's conjecture on the critical values of motivic L-functions. These results, as in the above mentioned works, are, in general, based on a nonvanishing hypothesis on certain archimedean integrals.

Contents

Date: November 26, 2008.
2000 *Mathematics Subject Classification.* 11F67 (11F70, 11F75, 22E55).

1. Introduction and statements of theorems

Let Π (respectively, Σ) be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ (respectively, $\mathrm{GL}_{n-1}(\mathbb{A})$); here $\mathbb{A}$ is the adèle ring of $\mathbb{Q}$. We assume the representations are such that $s = 1/2$ is critical for the Rankin–Selberg L-function attached to $\Pi \times \Sigma$. We prove an algebraicity result for $L(1/2, \Pi \times \Sigma)$. See Theorem 1.1. This is a generalization and refinement of the results of Harder [15], Kazhdan, Mazur and Schmidt [23], Mahnkopf [29], and Kasten and Schmidt [22]. Our result, as in the above mentioned works, is, in general, based on a nonvanishing hypothesis on certain archimedean integrals. We also prove a conjecture of Blasius and Panchishkin on twisted L-values in some cases using the period relations proved in our paper with Shahidi [33]; see Theorem 1.2.

Let φ be a holomorphic cusp form of weight k. We consider twisted odd symmetric power L-functions $L(s, \mathrm{Sym}^{2n-1}\varphi, \xi)$, where ξ is any Dirichlet character. Using the above result on Rankin–Selberg L-functions, we prove algebraicity results for certain critical values of $L(s, \mathrm{Sym}^{2n-1}\varphi, \xi)$ when $n \leq 4$. See Theorem 1.3. For $n = 1$ this is a classical theorem due to Shimura [38]; indeed, in this case, our theorem boils down to Harder's proof [15] of Shimura's theorem. For $n = 2$, our proof may be regarded as a new proof of the result of Garrett and Harris [11] on symmetric cube L-functions. Our theorem is new for the fifth and seventh symmetric power L-functions. Assuming Langlands' functoriality one can prove similar algebraicity results for any odd symmetric power L-function.

We now describe the theorems proved in this paper in greater detail, toward which we need some notation. Given a regular algebraic cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A})$ one knows (from Clozel [6]) that there is a pure dominant integral weight μ such that Π has a nontrivial contribution to the cohomology of some locally symmetric space of GL_n with coefficients coming from the dual of the finite dimensional representation with highest weight μ. We denote this as $\Pi \in \mathrm{Coh}(G_n, \mu^\vee)$, for $\mu \in X_0^+(T_n)$, where T_n is the diagonal torus of $G_n = \mathrm{GL}_n$. Under this assumption on Π, one knows that its rationality field $\mathbb{Q}(\Pi)$ is a number field, and that Π is defined over this number field. It is further known that the Whittaker model of Π carries a $\mathbb{Q}(\Pi)$-structure, and likewise, a suitable cohomology space also carries a rational structure. One defines a period $p^\epsilon(\Pi)$ by comparing these rational structures; here ϵ is a sign which can be arbitrary if n is even, and is uniquely determined by Π if n is odd. We briefly review the definition of these periods in 3.2.1, and refer the reader to [33] for more details. We now state one of the main theorems of this paper:

Theorem 1.1. *Let Π (resp., Σ) be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ (resp., $\mathrm{GL}_{n-1}(\mathbb{A})$). Let $\mu \in X_0^+(T_n)$ be such that $\Pi \in \mathrm{Coh}(G_n, \mu^\vee)$, and let $\lambda \in X_0^+(T_{n-1})$ be such that $\Sigma \in \mathrm{Coh}(G_{n-1}, \lambda^\vee)$. Assume that $\mu^\vee \succ \lambda$ (see §2 for the definition and a consequence of this condition). Assume also that $s = 1/2$ is critical for $L_f(s, \Pi \times \Sigma)$ which is the finite part of the Rankin–Selberg L-function attached to the pair (Π, Σ). There exists canonical signs $\epsilon, \eta \in \{\pm\}$ attached to the pair (Π, Σ); there exists nonzero complex numbers $p^\epsilon(\Pi)$, $p^\eta(\Sigma)$, and assuming the validity of Hypothesis 3.10 there exists a nonzero complex number $p_\infty(\mu, \lambda)$, such that for any $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have*

$$\sigma\left(\frac{L_f(1/2, \Pi \times \Sigma)}{p^\epsilon(\Pi) p^\eta(\Sigma) \mathcal{G}(\omega_{\Sigma_f})\, p_\infty(\mu, \lambda)}\right) = \frac{L_f(1/2, \Pi^\sigma \times \Sigma^\sigma)}{p^\epsilon(\Pi^\sigma)\, p^\eta(\Sigma^\sigma)\, \mathcal{G}(\omega_{\Sigma_f^\sigma})\, p_\infty(\mu, \lambda)},$$

where $\mathcal{G}(\omega_{\Sigma_f})$ is the Gauss sum attached to the central character of Σ. In particular,

$$L_f(1/2, \Pi \times \Sigma) \ \sim_{\mathbb{Q}(\Pi,\Sigma)} \ p^{\epsilon}(\Pi)\, p^{\eta}(\Sigma)\, \mathcal{G}(\omega_{\Sigma_f})\, p_\infty(\mu, \lambda),$$

where, by $\sim_{\mathbb{Q}(\Pi,\Sigma)}$, we mean up to an element of the number field which is the compositum of the rationality fields $\mathbb{Q}(\Pi)$ and $\mathbb{Q}(\Sigma)$ of Π and Σ respectively.

The proof of the above theorem is based on a cohomological interpretation of the Rankin–Selberg zeta integral. That the Rankin–Selberg integral for $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ admits a cohomological interpretation has been observed by several people. See especially, Schmidt [36], Kazhdan, Mazur and Schmidt [23], Mahnkopf [28], [29], and Kasten and Schmidt [22]. However, for the application we have in mind, which is Deligne's conjecture for symmetric power L-functions, the above works are not suitable because of various assumptions made in those papers. We prove the above theorem while refining their techniques, especially those of Mahnkopf [29]. The refinements are of two kinds:

(1) We do not twist by a highly ramified character at places where Π or Σ is ramified as is done in [29]. Instead, we use the observation that local special values are suitably rational (Proposition 3.17), and the possibly transcendental part of a global L-function is already captured by partial L-functions.
(2) The above papers are tailored toward constructing p-adic L-functions, in view of which there is a certain unipotent averaging that they consider at a prime where everything else is unramified. We consider the usual Rankin–Selberg integrals without any such unipotent averaging. It is quite likely that our theorem above, plus a refinement of the period relations proved in our paper with Shahidi [33], can also be used to construct p-adic L-functions.

We briefly sketch the proof of Theorem 1.1. We make a very specific choice of Whittaker vectors for the two representations, and show that the Rankin–Selberg zeta integral of the cusp forms corresponding to these vectors, at $s = 1/2$, can be interpreted as a pairing between certain cohomology classes. We choose a Whittaker vector w_{Π_f} for the finite part Π_f, and let ϕ_Π be the cusp form corresponding to $w_{\Pi_f} \otimes w_{\Pi_\infty}$, where w_{Π_∞} is a Whittaker vector at infinity. Similarly, for a specific vector w_{Σ_f}, consider a cusp form ϕ_Σ. The Rankin–Selberg integral at $1/2$ of these cusp forms, denoted $I(1/2, \phi_\Pi, \phi_\Sigma)$, is, up to controllable quantities, the L-value we are interested in (Proposition 3.1). On the other hand, it may be interpreted as follows. To w_{Π_f} is attached a cuspidal cohomology class ϑ_Π in $H^{b_n}_{\mathrm{cusp}}(F_n, \mathcal{M}^\vee_\mu)$, where b_n is the bottom degree of the cuspidal range for GL_n, F_n is a tentative notation for a locally symmetric space associated to GL_n, and $\mathcal{M}^\vee_\mu$ is the sheaf on F_n corresponding to the dual of the finite dimensional representation M_μ with highest weight μ. Working with the dual of M_μ is only for convenience. Similarly, we have $\vartheta_\Sigma \in H^{b_{n-1}}_{\mathrm{cusp}}(F_{n-1}, \mathcal{M}^\vee_\lambda)$. The hypothesis $\mu^\vee \succ \lambda$ implies that there is a canonical G_{n-1} pairing $M^\vee_\mu \times M^\vee_\lambda \to \mathbb{Q}$. The natural embedding $\mathrm{GL}_{n-1} \to \mathrm{GL}_n$ induces a proper map $\iota : F_{n-1} \to F_n$. We consider the wedge product $\vartheta_\Sigma \wedge \iota^*\vartheta_\Pi$, and observe that this happens to be a top-degree form on F_{n-1} because $b_{n-1} + b_n = \dim(F_{n-1})$; this numerical coincidence is at the heart of the proof. Integrating the top degree form over all of F_{n-1} gives, after unravelling the definitions and using the calculation of the Rankin–Selberg integrals mentioned above, nothing but $L_f(1/2, \Pi \times \Sigma)\langle[\Sigma_\infty], [\Pi_\infty]\rangle$. This is the content of the *main identity* proved in Theorem 3.12.

The quantity $\langle [\Sigma_\infty], [\Pi_\infty] \rangle$, which depends only on the representations at infinity, is a linear combination of Rankin–Selberg integrals for 'cohomological vectors'. One expects that it is nonzero. We have not attempted a proof of this nonvanishing hypothesis, and so we need to assume its validity. The proof of Theorem 1.1 follows since we can control algebraicity properties of the pairing of the classes ϑ_Π and ϑ_Σ.

We now come to the second main theorem of this paper, which is to understand the behaviour of L-values under twisting by characters. We refer the reader to our papers with Shahidi [32] and [33] for motivational background for such results. We note that results of this kind are predicted by the results and conjectures of Blasius [3] and Panchishkin [31], both of whom independently calculated the behaviour of Deligne's periods attached to a motive upon twisting by Artin motives. Our second theorem is:

Theorem 1.2. *Let Π and Σ be as in Theorem 1.1. Let ξ be an even Dirichlet character which we identify with the corresponding Hecke character of $\mathbb{Q}$. We have*

$$L_f(1/2, (\Pi \otimes \xi) \times \Sigma) \sim_{\mathbb{Q}(\Pi,\Sigma,\xi)} \mathcal{G}(\xi_f)^{n(n-1)/2} L_f(1/2, \Pi \times \Sigma),$$

where, by $\sim_{\mathbb{Q}(\Pi,\Sigma,\xi)}$, we mean up to an element of the number field $\mathbb{Q}(\Pi, \Sigma, \xi)$ which is the compositum of the rationality fields $\mathbb{Q}(\Pi)$, $\mathbb{Q}(\Sigma)$ and $\mathbb{Q}(\xi)$ of Π, Σ, and ξ respectively. Moreover, if $L_f(1/2, \Pi \times \Sigma) \neq 0$, then for any $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have

$$\sigma\left(\frac{L_f(1/2, (\Pi \otimes \xi) \times \Sigma)}{\mathcal{G}(\xi_f)^{n(n-1)/2}\, L_f(1/2, \Pi \times \Sigma)}\right) = \frac{L_f(1/2, (\Pi^\sigma \otimes \xi^\sigma) \times \Sigma^\sigma)}{\mathcal{G}(\xi_f^\sigma)^{n(n-1)/2}\, L_f(1/2, \Pi^\sigma \times \Sigma^\sigma)}.$$

We remark that our proof of Theorem 1.2 uses Theorem 1.1, and so is subject to the assumption made in Hypothesis 3.10.

We now describe an application of Theorem 1.1 to the special values of symmetric power L-functions. Let φ be a primitive holomorphic cusp form on the upper half plane of weight k, for $\Gamma_0(N)$, with nebentypus character ω. We denote this as $\varphi \in S_k(N, \omega)_{\mathrm{prim}}$. For any integer $r \geq 1$, consider the r-th symmetric power L-function $L(s, \mathrm{Sym}^r \varphi, \xi)$ attached to φ, twisted by a Dirichlet character ξ. The sign of ξ is defined as $\epsilon_\xi = \xi(-1)$. (We will think of ξ as a Hecke character of $\mathbb{Q}$.) Our final theorem in this paper gives an algebraicity theorem for certain critical values of such L-functions when r is an odd integer ≤ 7.

Theorem 1.3. *Let $\varphi \in S_k(N, \omega)_{\mathrm{prim}}$, n a positive integer ≤ 4, and ξ a Dirichlet character. Let m be the critical integer for $L_f(s, \mathrm{Sym}^{2n-1}(\varphi), \xi)$ given by:*

1. *If k is even, then we assume $k \geq 4$ and let $m = ((2n-1)(k-1)+3)/2$.*
2. *If k is odd, then we assume $k \geq 3$ and let $m = ((2n-1)(k-1)+2)/2$.*

There exists nonzero complex numbers $p^\epsilon(\varphi, 2n-1)$ depending on the form φ, the integer n, and a sign $\epsilon \in \{\pm\}$, and there exists a nonzero complex number $p(m,k)$ depending on the critical point m and the weight k, such that for any $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have

$$\sigma\left(\frac{L_f(m, \mathrm{Sym}^{2n-1}(\varphi), \xi)}{p^{\epsilon_\xi}(\varphi, 2n-1)\, p(m,k)\, \mathcal{G}(\xi_f)^n}\right) = \frac{L_f(m, \mathrm{Sym}^{2n-1}(\varphi^\sigma), \xi^\sigma)}{p^{\epsilon_\xi}(\varphi^\sigma, 2n-1)\, p(m,k)\, \mathcal{G}(\xi_f^\sigma)^n}.$$

In particular,

$$L_f(m, \mathrm{Sym}^{2n-1}(\varphi), \xi) \ \sim_{\mathbb{Q}(\varphi,\xi)} \ p^{\epsilon_\xi}(\varphi, 2n-1)\, p(m,k)\, \mathcal{G}(\xi_f)^n,$$

where, by $\sim_{\mathbb{Q}(\varphi,\xi)}$, we mean up to an element of the number field generated by the Fourier coefficients of φ and the values of ξ.

Further, if we assume Langlands' functoriality, in as much as assuming that the transfer of automorphic representations holds for the L-homomorphism $\mathrm{Sym}^l : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_{l+1}(\mathbb{C})$ for all integers $l \geq 1$, then the above statements about critical values holds for all odd positive integers $2n-1$.

We remark that our proof of Theorem 1.3 uses Theorem 1.1, and so is subject to the assumption made in Hypothesis 3.10. Let $\pi(\varphi)$ be the cuspidal automorphic representation attached to φ, and let $\mathrm{Sym}^r(\pi(\varphi))$ denote the r-th symmetric power transfer; it is known to exist for $r \leq 4$ by the work of Gelbart and Jacquet [12], Kim and Shahidi [25], and Kim [24]. The proof of Theorem 1.3 is obtained by recursively applying Theorem 1.1 to the pair $(\mathrm{Sym}^n(\pi(\varphi)), \mathrm{Sym}^{n-1}(\pi(\varphi)))$, up to appropriate twisting (Proposition 5.4). The critical point m that we consider is on the right edge of symmetry when k is odd, and is one unit to the right of the center of symmetry when k is even. The quantity $p^\epsilon(\varphi, 2n-1)$ is a combination of periods attached to $\mathrm{Sym}^r(\pi(\varphi))$ for $r \leq n$, and the quantity $p(m,k)$ is a combination of some of the $p_\infty(\mu,\lambda)$ that show up in Theorem 1.1. We expect that our results are compatible with Deligne's conjecture [9, §7.8], in view of which, we formulate Conjecture 5.15 relating the periods attached to the representations Π and Σ as above, and Deligne's periods $c^\pm(M)$, where M is the tensor product of the conjectural motives $M(\Pi)$ and $M(\Sigma)$.

Finally, we note that in this paper we have considered only one critical point for any given L-function. In joint work with Günter Harder, we are investigating the algebraicity properties of ratios of successive critical values for the Rankin–Selberg L-functions considered above. This will then give us algebraicity results for ratios of successive critical values for the odd symmetric power L-functions considered above. The results of this investigation will appear elsewhere. On an entirely different note, we mention the recent work of Gan, Gross and Prasad [10] on generalizations of the Gross-Prasad conjectures; they too are interested in the central critical value $L(1/2, \Pi \times \Sigma)$, albeit, from a different perspective.

Acknowledgements: It is a pleasure to thank Don Blasius, Günter Harder, Michael Harris, Paul Garrett, Ameya Pitale and Freydoon Shahidi for their interest and helpful discussions. I thank Jishnu Biswas and Vishwambar Pati for clarifying some topological details. Much of this work was carried out during a visit to the Max Planck Institute in 2008; I gratefully acknowledge their invitation and thank MPI for providing an excellent atmosphere.

2. Notations, conventions, and preliminaries

The algebraic group GL_n over $\mathbb{Q}$ will be denoted as G_n. Let $B_n = T_n N_n$ stand for the standard Borel subgroup of G_n of all upper triangular matrices, N_n the unipotent radical of B_n, and T_n the diagonal torus. The center of G_n will be denoted by Z_n. The identity element of G_n will be denoted 1_n.

We let $X^+(T_n)$ stand for the set of dominant (with respect to B_n) integral weights of T_n, and for $\mu \in X^+(T_n)$ we denote by M_μ the irreducible representation of $G_n(\mathbb{C})$ with highest weight μ. Note that M_μ is defined over $\mathbb{Q}$. Let $M_\mu^\vee$ denote the contragredient of M_μ and define the dual weight $\mu^\vee$ by $M_\mu^\vee = M_{\mu^\vee}$. We let $X_0^+(T_n)$ stand for the subset

of $X^+(T_n)$ consisting of pure weights [29, (3.1)]. If $\mu = (\mu_1, \dots, \mu_n) \in X^+(T_n)$ and $\lambda = (\lambda_1, \dots, \lambda_{n-1}) \in X^+(T_{n-1})$ then by $\mu \succ \lambda$ we mean the condition $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_n$, which ensures that M_λ appears in the restriction to G_{n-1} of M_μ; in fact it appears with multiplicity one.

We let $\mathbb{A}$ stand for the adèle ring of $\mathbb{Q}$, and $\mathbb{A}_f$ the ring of finite adèles. Following Borel–Jacquet [4, §4.6], we say an irreducible representation of $G_n(\mathbb{A})$ is automorphic if it is isomorphic to an irreducible subquotient of the representation of $G_n(\mathbb{A})$ on its space of automorphic forms. We say an automorphic representation is cuspidal if it is a subrepresentation of the representation of $G_n(\mathbb{A})$ on the space of cusp forms $\mathcal{A}_{\mathrm{cusp}}(G_n(\mathbb{Q})\backslash G_n(\mathbb{A}))$. The subspace of cusp forms realizing Π will be denoted V_Π. For an automorphic representation Π of $G_n(\mathbb{A})$, we have $\Pi = \Pi_\infty \otimes \Pi_f$, where Π_∞ is a representation of $G_{n,\infty} = G_n(\mathbb{R})$, and $\Pi_f = \otimes_{v \neq \infty} \Pi_v$ is a representation of $G_n(\mathbb{A}_f)$. The central character of any irreducible representation Θ will be denoted ω_Θ. The finite part of a global L-function is denoted $L_f(s, \Pi)$, and for any place v the local L-factor at v is denoted $L(s, \Pi_v)$.

We will let $K_{n,\infty}$ stand for $\mathrm{O}(n)Z_n(\mathbb{R})$; it is the thickening of the maximal compact subgroup of $G_{n,\infty}$ by the center $Z_{n,\infty}$. Let $K^0_{n,\infty}$ be the topological connected component of $K_{n,\infty}$. For any group $\mathfrak{G}$ we will let $\pi_0(\mathfrak{G})$ stand for the group of connected components. We will identify $\pi_0(G_n) = \pi_0(K_{n,\infty}) \simeq \{\pm 1\} = \{\pm\}$. Note that $\delta_n = \mathrm{diag}(-1, 1, \dots, 1)$ represents the nontrivial element in $\pi_0(K_{n,\infty})$, and if n is odd, the element -1_n also represents this nontrivial element. We will further identify $\pi_0(K_{n,\infty})$ with its character group $\pi_0(K_{n,\infty})\widehat{}$. Let $K^1_{n,\infty} = \mathrm{SO}(n)$.

Let $\iota : G_{n-1} \to G_n$ be the map $g \mapsto \left(\begin{smallmatrix} g & \\ & 1 \end{smallmatrix}\right)$. Then ι induces a map at the level of local and global groups, and between appropriate symmetric spaces of G_{n-1} and G_n, all of which will also be denoted by ι again; we hope that this will cause no confusion. The pullback (of a subset, a function, a differential form, or a cohomology class) via ι will be denoted ι^*.

Fix a global measure dg on $G_n(\mathbb{A})$ which is a product of local measures dg_v. The local measures are normalized as follows: for a finite place v, if $\mathcal{O}_v$ is the ring of integers of $\mathbb{Q}_v$, then we assume that $\mathrm{vol}(G_n(\mathcal{O}_v)) = 1$; and at infinity assume that $\mathrm{vol}(K^1_{n,\infty}) = 1$.

For a Dirichlet character χ modulo an integer N, following Shimura [37], we define its Gauss sum $\mathfrak{g}(\chi)$ as the Gauss sum of its associated primitive character, say χ_0 of conductor c, where $\mathfrak{g}(\chi_0) = \sum_{a=0}^{c-1} \chi_0(a) e^{2\pi i a/c}$. For a Hecke character ξ of $\mathbb{Q}$, by which we mean a continuous homomorphism $\xi : \mathbb{Q}^* \backslash \mathbb{A}^\times \to \mathbb{C}^*$, following Weil [40, Chapter VII, §7], we define the Gauss sum of ξ as follows: We let $\mathfrak{c}$ stand for the conductor ideal of ξ_f. We fix, once and for all, an additive character ψ of $\mathbb{Q}\backslash\mathbb{A}$, as in Tate's thesis, namely, $\psi(x) = e^{2\pi i \Lambda(x)}$ with the Λ as defined in [39, §4.1]. Let $y = (y_v)_{v \neq \infty} \in \mathbb{A}_f^\times$ be such that $\mathrm{ord}_v(y_v) = -\mathrm{ord}_v(\mathfrak{c})$. The Gauss sum of ξ is defined as $\mathcal{G}(\xi_f, \psi_f, y) = \prod_{v \neq \infty} \mathcal{G}(\xi_v, \psi_v, y_v)$ where the local Gauss sum $\mathcal{G}(\xi_v, \psi_v, y_v)$ is defined as

$$\mathcal{G}(\xi_v, \psi_v, y_v) = \int_{\mathcal{O}_v^\times} \xi_v(u_v)^{-1} \psi_v(y_v u_v)\, du_v.$$

For almost all v, where everything in sight is unramified, we have $\mathcal{G}(\xi_v, \psi_v, y_v) = 1$, and for all v we have $\mathcal{G}(\xi_v, \psi_v, y_v) \neq 0$. (See, for example, Godement [13, Eqn. 1.22].) Note that, unlike Weil, we do not normalize the Gauss sum to make it have absolute value one and we do not have any factor at infinity. Suppressing the dependence on ψ and y, we

denote $\mathcal{G}(\xi_f, \psi_f, y)$ simply by $\mathcal{G}(\xi_f)$. To have the functional equations of the L-functions of a Dirichlet character χ and the corresponding Hecke character ξ to *look the same* we need the Gauss sums to be defined as above; compare Neukirch [30, Chapter VII, Theorem 2.8] with Weil [40, Chapter VII, Theorem 5].

In our paper with Shahidi [33] we defined the Gauss sum $\gamma(\xi_f)$ of a Hecke character ξ as $\mathcal{G}(\xi_f^{-1})$. Since this article crucially uses the results of [33] it is helpful to record the following details that we will repeatedly use: Lemma 4.3 of [33] now reads as

$$\sigma(\xi_f(t_\sigma)) = \sigma(\mathcal{G}(\xi_f))/\mathcal{G}(\xi_f^\sigma), \tag{2.1}$$

and Theorem 4.1(1) of [33] now reads as

$$\sigma\left(\frac{p^{\epsilon\cdot\epsilon_\xi}(\Pi_f \otimes \xi_f)}{\mathcal{G}(\xi_f)^{n(n-1)/2}\, p^\epsilon(\Pi_f)}\right) = \left(\frac{p^{\epsilon\cdot\epsilon_{\xi^\sigma}}(\Pi_f^\sigma \otimes \xi_f^\sigma)}{\mathcal{G}(\xi_f^\sigma)^{n(n-1)/2}\, p^\epsilon(\Pi_f^\sigma)}\right), \tag{2.2}$$

where Π is a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ and ξ is an algebraic Hecke character of $\mathbb{Q}$.

3. Rankin-Selberg L-functions for $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$

3.1. **The global integral.**

3.1.1. We consider the Rankin–Selberg zeta integrals for $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$. (See the works of Jacquet, Piatetski-Shapiro and Shalika [19], [21]. We roughly follow the notation in Cogdell's expository article [7].) Let Π (resp., Σ) be a cuspidal automorphic representation of $G_n(\mathbb{A})$ (resp., $G_{n-1}(\mathbb{A})$). Let $\phi \in V_\Pi$ and $\phi' \in V_\Sigma$ be cusp forms. The zeta integral we are interested in is given by

$$I(s, \phi, \phi') = \int_{G_{n-1}(\mathbb{Q})\backslash G_{n-1}(\mathbb{A})} \phi(\iota(g))\phi'(g)|\det(g)|^{s-1/2}\, dg.$$

Since the cusp forms ϕ and ϕ' are rapidly decreasing, the above integral converges for all $s \in \mathbb{C}$. Suppose that $w \in W(\Pi, \psi)$ and $w' \in W(\Sigma, \psi^{-1})$ are global Whittaker functions corresponding to ϕ and ϕ', respectively; recall that ψ is a nontrivial additive character $\mathbb{Q}\backslash\mathbb{A}$. After the usual unfolding, one has

$$I(s, \phi, \phi') = \Psi(s, w, w') := \int_{N_{n-1}(\mathbb{A})\backslash G_{n-1}(\mathbb{A})} w(\iota(g))w'(g)|\det(g)|^{s-1/2}\, dg.$$

The integral $\Psi(s, w, w')$ converges for $\mathrm{Re}(s) \gg 0$. Let $w = \otimes w_v$ and $w' = \otimes w_v'$, then $\Psi(s, w, w') := \otimes \Psi_v(s, w_v, w_v')$ for $\mathrm{Re}(s) \gg 0$, where the local integral Ψ_v is given by a similar formula. Recall that the local integral $\Psi_v(s, w_v, w_v')$ converges for $\mathrm{Re}(s) \gg 0$ and has a meromorphic continuation to all of $\mathbb{C}$; see [7, Proposition 6.2] for $v < \infty$, and for $v = \infty$ see [8, Theorem 1.2(i)]. We will choose the local Whittaker functions carefully so that the integral $I(1/2, \phi, \phi')$ computes the special value $L_f(1/2, \Pi \times \Sigma)$ up to quantities which are under control, in the sense that they will be $\mathrm{Aut}(\mathbb{C})$-equivariant. Before making this choice of vectors, we review some ingredients.

3.1.2. *Action of* $\mathrm{Aut}(\mathbb{C})$ *on Whittaker models.* Consider the cyclotomic character

$$\begin{array}{ccccccc} \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) & \to & \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \to & \mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) & \to & \widehat{\mathbb{Z}}^\times \simeq \prod_p \mathbb{Z}_p^\times \\ \sigma & \mapsto & \sigma|_{\overline{\mathbb{Q}}} & \mapsto & \sigma|_{\mathbb{Q}(\mu_\infty)} & \mapsto & t_\sigma \end{array}$$

The element t_σ at the end can be thought of as an element of $\mathbb{A}_f^\times = \mathbb{I}_f$. Let $t_{\sigma,n}$ denote the diagonal matrix $\mathrm{diag}(t_\sigma^{-(n-1)}, t_\sigma^{-(n-2)}, \dots, 1)$ regarded as an element of $\mathrm{GL}_n(\mathbb{A}_f)$. For $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $w \in W(\Pi_f, \psi_f)$, define the function ${}^\sigma w$ by

$${}^\sigma w(g_f) = \sigma(w(t_{\sigma,n} g_f))$$

for all $g_f \in \mathrm{GL}_n(\mathbb{A}_f)$. Note that this action makes sense locally, by replacing t_σ by $t_{\sigma,v}$. Further, if Π_v is unramified, then the spherical vector is mapped to the spherical vector under σ. This makes the local and global actions compatible. For more details, see [33, §3.2]. (See also §3.4 where we discuss other possible actions of $\mathrm{Aut}(\mathbb{C})$ on Whittaker models.)

3.1.3. *Normalized new vectors.* We review some details about local new (or essential) vectors [20]. Just for this paragraph, let F be a non-archimedean local field, $\mathcal{O}_F$ the ring of integers of F, and $\mathcal{P}_F$ the maximal ideal of $\mathcal{O}_F$. Let (π, V) be an irreducible admissible generic representation of $\mathrm{GL}_n(F)$. Let $K_n(m)$ be the 'mirahoric subgroup' of $\mathrm{GL}_n(\mathcal{O}_F)$ consisting of all matrices whose last row is congruent to $(0, \dots, 0, *)$ modulo $\mathcal{P}_F^m$. Let $V_m := \{v \in V \mid \pi(k)v = \omega_\pi(k_{n,n})v, \forall k \in K_n(m)\}$. Let $\mathfrak{f}(\pi)$ be the least non-negative integer m for which $V_m \neq (0)$. One knows that $\mathfrak{f}(\pi)$ is the conductor of π (in the sense of epsilon factors), and that $V_{\mathfrak{f}(\pi)}$ is one-dimensional. Any vector in $V_{\mathfrak{f}(\pi)}$ is called a *new vector* of π. Fix a nontrivial additive character ψ of F, and assume that $V = W(\pi, \psi)$ is the Whittaker model for π. If π is unramified, i.e., $\mathfrak{f}(\pi) = 0$, then we fix a specific new vector called the *spherical vector*, which we denote w_π^{sp}, normalized such that $w_\pi^{\mathrm{sp}}(1_n) = 1$. More generally, for any π, amongst all new vectors, there is a distinguished vector, called the *essential vector*, which we denote as w_π^{ess}, characterized by the property that for any irreducible unramified generic representation ρ of $\mathrm{GL}_{n-1}(F)$ one has

$$\Psi(s, w_\pi^{\mathrm{ess}}, w_\rho^{\mathrm{sp}}) = \int_{N_{n-1}(F)\backslash G_{n-1}(F)} w_\pi^{\mathrm{ess}}(\iota(g)) w_\rho^{\mathrm{sp}}(g) |\det(g)|^{s-1/2}\, dg = L(s, \pi \times \rho).$$

We note that if π is unramified then $w_\pi^{\mathrm{ess}} = w_\pi^{\mathrm{sp}}$. Although the essential vector has the above nice analytic property, it does not, in general, have good arithmetic properties in the sense that essential vectors are not $\mathrm{Aut}(\mathbb{C})$-equivariant. For this equivariance, following Mahnkopf, using [29, Lemma 1.3.2], we fix the following normalization. This lemma says that given π there exists $t_\pi \in T_n(F)$ such that a new vector for π is nonvanishing on t_π. Note that necessarily $t_\pi \in T_n^+(F)$, i.e., if $t_\pi = \mathrm{diag}(t_1, t_2, \dots, t_n)$ then $t_i t_{i+1}^{-1} \in \mathcal{O}_F$ for all $1 \leq i \leq n-1$. We let w_π^0 be the new vector normalized such that $w_\pi^0(t_\pi) = 1$. If π is unramified then we may and will take $t_\pi = 1_n$, and so $w_\pi^0 = w_\pi^{\mathrm{ess}} = w_\pi^{\mathrm{sp}}$. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$ we may and will take $t_{\pi^\sigma} = t_\pi$. Then it is easy to see that ${}^\sigma w_\pi^0 = w_{\pi^\sigma}^0$. We define the scalar $c_\pi \in \mathbb{C}^*$ by $w_\pi^0 = c_\pi w_\pi^{\mathrm{ess}}$, i.e., $c_\pi = w_\pi^{\mathrm{ess}}(t_\pi)^{-1}$.

3.1.4. *Choice of Whittaker vectors and cusp forms.* We now go back to global notation and choose global Whittaker vectors $w_\Pi = \otimes_v w_{\Pi,v} \in W(\Pi, \psi)$ and $w_\Sigma = \otimes_v w_{\Sigma,v} \in W(\Sigma, \psi^{-1})$ as follows. Let S_Σ be the set of finite places v where Σ_v is unramified.

(1) If $v \notin S_\Sigma \cup \{\infty\}$, we let $w_{\Pi,v} = w^0_{\Pi_v}$, and $w_{\Sigma,v} = w^{\rm sp}_{\Sigma_v}$.
(2) If $v \in S_\Sigma$, we let $w_{\Sigma,v} = w^0_{\Sigma_v}$, and let $w_{\Pi,v}$ be the unique Whittaker function whose restriction to $G_{n-1}(\mathbb{Q}_v)$ is supported on $N_{n-1}(\mathbb{Q}_v)t_{\Sigma_v}K_{n-1}(\mathfrak{f}(\Sigma_v))$, and on this double coset it is given by $w_{\Pi,v}(ut_{\Sigma_v}k) = \psi(u)\omega^{-1}_{\Sigma_v}(k_{n-1,n-1})$, for all $u \in N_{n-1}(\mathbb{Q}_v)$ and for all $k \in K_{n-1}(\mathfrak{f}(\Sigma_v))$.
(3) If $v = \infty$, we let $w_{\Pi,\infty}$ and $w_{\Sigma,\infty}$ be arbitrary nonzero vectors. (Later, these will be cohomological vectors.)

Let $w_{\Pi_f} = \otimes_{v\neq\infty} w_{\Pi,v}$ and $w_\Pi = w_{\Pi_\infty} \otimes w_{\Pi_f}$. Similarly, let $w_{\Sigma_f} = \otimes_{v\neq\infty} w_{\Sigma,v}$ and $w_\Sigma = w_{\Sigma_\infty} \otimes w_{\Sigma_f}$. Let ϕ_Π (resp., ϕ_Σ) be the cusp form corresponding to w_Π (resp., w_Σ).

3.1.5. *Rankin–Selberg L-functions.*

Proposition 3.1. *We have*

$$I(1/2, \phi_\Pi, \phi_\Sigma) \;=\; \frac{\Psi_\infty(1/2, w_{\Pi_\infty}, w_{\Sigma_\infty})\,\mathrm{vol}(\Sigma)\,\prod_{v\notin S_\Sigma\cup\{\infty\}} c_{\Pi_v}}{\prod_{v\in S_\Sigma} L(1/2, \Pi_v \times \Sigma_v)} L_f(1/2, \Pi\times\Sigma),$$

where $\mathrm{vol}(\Sigma) = \prod_{v\in S_\Sigma} \mathrm{vol}(K_{n-1}(\mathfrak{f}(\Sigma_v))) \in \mathbb{Q}^*$.

Proof.

$$\begin{aligned} I(s, \phi_\Pi, \phi_\Sigma) &= \int_{G_{n-1}(\mathbb{Q})\backslash G_{n-1}(\mathbb{A})} \phi_\Pi(\iota(g))\phi_\Sigma(g)|\det(g)|^{s-1/2}\,dg, \quad (\forall s\in\mathbb{C}) \\ &= \int_{N_{n-1}(\mathbb{A})\backslash G_{n-1}(\mathbb{A})} w_\Pi(\iota(g))w_\Sigma(g)|\det(g)|^{s-1/2}\,dg, \quad (\mathrm{Re}(s)\gg 0) \\ &= \prod_v \int_{N_{n-1}(\mathbb{Q}_v)\backslash G_{n-1}(\mathbb{Q}_v)} w_{\Pi_v}(\iota(g_v))w_{\Sigma_v}(g_v)|\det(g_v)|^{s-1/2}\,dg_v \\ &= \Psi_\infty(s, w_{\Pi_\infty}, w_{\Sigma_\infty}) \prod_{v\notin S_\Sigma\cup\{\infty\}} c_{\Pi_v} L(s, \Pi_v\times\Sigma_v) \prod_{v\in S_\Sigma} \mathrm{vol}(K_{n-1}(\mathfrak{f}(\Sigma_v))). \end{aligned}$$

The last equality is because of our specific choice of Whittaker vectors. Multiplying and dividing by the local factors for $v \in S_\Sigma$ we get

$$I(s, \phi_\Pi, \phi_\Sigma) = \frac{\Psi_\infty(s, w_{\Pi_\infty}, w_{\Sigma_\infty})\,\mathrm{vol}(\Sigma)\,\prod_{v\notin S_\Sigma\cup\{\infty\}} c_{\Pi_v}}{\prod_{v\in S_\Sigma} L(s, \Pi_v\times\Sigma_v)} L_f(s, \Pi\times\Sigma), \quad (\mathrm{Re}(s)\gg 0).$$

The left hand side is defined for all s, and the right hand side has a meromorphic continuation to all of $\mathbb{C}$. Hence we get equality at $s = 1/2$. Since $c_{\Pi_v} = 1$ if Π_v is unramified, the product $\prod_{v\notin S_\Sigma\cup\{\infty\}} c_{\Pi_v}$ is really a finite product. □

3.2. **Cohomological interpretation of the integral.** We interpret the Rankin–Selberg integral $I(1/2, \phi_\Pi, \phi_\Sigma)$ in terms of Poincaré duality. More precisely, the vector w_{Π_f} will correspond to a cohomology class ϑ_Π in degree b_n (the bottom degree of the cuspidal range for G_n) on a locally symmetric space tentatively denoted F_n for GL_n, and similarly w_{Σ_f} will correspond to a class ϑ_Σ in degree b_{n-1} on F_{n-1}. These classes, after dividing by certain periods, have good rationality properties. We pull back ϑ_Π along the proper map $\iota : F_{n-1} \to F_n$, and wedge (or cup) with ϑ_Σ, to give a top degree class on F_{n-1}. It is of top degree because $b_n + b_{n-1} = \dim(F_{n-1})$; this numerical coincidence is at the heart of other

works on Rankin–Selberg L-functions. (See, for example, Kazhdan, Mazur and Schmidt [23, §1] or Mahnkopf [29, p. 616].) Integrating this form on F_{n-1}, which indeed is the Rankin–Selberg integral of the previous section, is nothing but applying a linear functional to cohomology in top degree, and the point is that this functional is that obtained from pairing with a certain cycle (constructed as in Mahnkopf [29, 5.1.1], which in turn is a generalization of Harder's construction [15] for GL_2). Interpreting the integral, and hence the special values of L-functions, as a cohomological pairing permits us to study arithmetic properties of the special values, since this pairing is Galois equivariant. We now make all this precise.

3.2.1. *The periods.* We assume that the reader is familiar with our paper with Shahidi [33], and especially the definition of the periods attached to regular algebraic cuspidal representations. We review the very basic ingredients here, and refer the reader to [33] for all finer details. See especially [33, Definition/Proposition 3.3]. We also use the same notation as in that paper, with just one exception that we mention in the next paragraph.

Assume now that the cuspidal representation Π (resp., Σ) is regular and algebraic. A consequence is that there is a weight $\mu \in X^+(T_n)$ (resp., $\lambda \in X^+(T_{n-1})$) such that $\Pi \in \mathrm{Coh}(G_n, \mu^\vee)$ (resp., $\Sigma \in \mathrm{Coh}(G_{n-1}, \lambda^\vee)$). The weight μ is a dominant integral weight which is *pure* (by [6, Lemme de pureté 4.9]), i.e., if $\mu = (\mu_1, \dots, \mu_n)$, then there is an integer $\mathrm{wt}(\mu)$ such that $\mu_i + \mu_{n-i+1} = \mathrm{wt}(\mu)$. We will denote by $X_0^+(T_n)$ the set of dominant integral pure weights for T_n. Similarly, $\lambda \in X_0^+(T_{n-1})$. Let $\epsilon \in \{\pm\} \simeq (K_{n,\infty}/K_{n,\infty}^0)\widehat{}$ be a sign, which can be arbitrary if n is even, and is uniquely determined by Π if n is odd. (If n is odd then $\epsilon = \omega_{\Pi_\infty}(-1)(-1)^{\mathrm{wt}(\mu)/2}$, which is the central character of $\Pi_\infty \otimes M_\mu^\vee$ at -1.) Such an ϵ is called a permissible sign for Π. We define $b_n = n^2/4$ if n is even, and $b_n = (n^2-1)/4$ if n is odd. We have a map

$$\mathcal{F}_{\Pi_f,\epsilon,[\Pi_\infty]} : W(\Pi_f) \to H^{b_n}(\mathfrak{g}_\infty, K_\infty^0; V_\Pi \otimes M_\mu^\vee)(\epsilon).$$

We note that difference in notation mentioned above: a choice of generator for the one-dimensional $\mathbb{C}$-vector space $H^{b_n}(\mathfrak{g}_\infty, K_\infty^0; \Pi_\infty \otimes M_\mu^\vee)(\epsilon)$ which was denoted $\mathbf{w}_\infty$ in [33], will be denoted by $[\Pi_\infty]$ in this paper. The map $\mathcal{F}_{\Pi_f,\epsilon,[\Pi_\infty]}$ is a $G_n(\mathbb{A}_f)$-equivariant map between irreducible modules, both of which have $\mathbb{Q}(\Pi)$-rational structures that are unique up to homotheties. For the action of $\mathrm{Aut}(\mathbb{C})$ and the rational structure on the Whittaker model $W(\Pi_f)$ see [33, §3.2], and on $H^{b_n}(\mathfrak{g}_\infty, K_\infty^0; V_\Pi \otimes M_\mu^\vee)(\epsilon)$ see [33, §3.3]. The period $p^\epsilon(\Pi)$ is defined by requiring the normalized map

$$\mathcal{F}^0_{\Pi_f,\epsilon,[\Pi_\infty]} := p^\epsilon(\Pi)^{-1}\mathcal{F}_{\Pi_f,\epsilon,[\Pi_\infty]}$$

to be $\mathrm{Aut}(\mathbb{C})$-equivariant, i.e., for all $\sigma \in \mathrm{Aut}(\mathbb{C})$ one has

$$\sigma \circ \mathcal{F}^0_{\Pi_f,\epsilon,[\Pi_\infty]} = \mathcal{F}^0_{{}^\sigma\Pi_f,\epsilon,[{}^\sigma\Pi_\infty]} \circ \sigma.$$

3.2.2. *The cohomology classes.* We now define the classes attached to the global Whittaker vectors w_{Π_f} and w_{Σ_f}:

$$\vartheta_{\Pi,\epsilon} := \mathcal{F}_{\Pi_f,\epsilon,[\Pi_\infty]}(w_{\Pi_f}), \quad \vartheta^0_{\Pi,\epsilon} := \mathcal{F}^0_{\Pi_f,\epsilon,[\Pi_\infty]}(w_{\Pi_f}) = p^\epsilon(\Pi)^{-1}\vartheta_{\Pi,\epsilon}, \tag{3.2}$$

and similarly,

$$\vartheta_{\Sigma,\eta} := \mathcal{F}_{\Sigma_f,\eta,[\Sigma_\infty]}(w_{\Sigma_f}), \quad \vartheta^0_{\Sigma,\eta} := \mathcal{F}^0_{\Sigma_f,\eta,[\Sigma_\infty]}(w_{\Sigma_f}) = p^\eta(\Sigma)^{-1}\vartheta_{\Sigma,\eta}. \tag{3.3}$$

Let K_f be an open compact subgroup of $G_n(\mathbb{A}_f)$ which fixes w_{Π_f} and such that ι^*K_f fixes w_{Σ_f}. Note that $\vartheta_{\Pi,\epsilon}$, which, by definition, lies in $H^{b_n}(\mathfrak{g}_\infty, K^0_\infty; V_\Pi \otimes M^\vee_\mu)(\epsilon)$, actually lies in $H^{b_n}(\mathfrak{g}_\infty, K^0_\infty; V_\Pi^{K_f} \otimes M^\vee_\mu)(\epsilon)$, and by the same token, $\vartheta_{\Sigma,\eta} \in H^{b_{n-1}}(\mathfrak{g}_\infty, K^0_\infty; V_\Sigma^{\iota^*K_f} \otimes M^\vee_\lambda)(\eta)$. Consider the manifolds:

$$\begin{aligned} S_n(K_f) &:= G_n(\mathbb{Q})\backslash G_n(\mathbb{A})/K^0_{n,\infty}K_f, \\ S_{n-1}(\iota^*K_f) &:= G_{n-1}(\mathbb{Q})\backslash G_{n-1}(\mathbb{A})/K^0_{n-1,\infty}\iota^*K_f \end{aligned}$$

Via certain standard isomorphisms ([33, §3.3]) we may identify the class $\vartheta_{\Pi,\epsilon}$ as a class in $H^{b_n}_{\mathrm{cusp}}(S_n(K_f), \mathcal{M}^\vee_\mu)(\tilde{\Pi}_f)$ where $\tilde{\Pi}_f := \Pi_f \otimes \epsilon$ is a representation of $G_n(\mathbb{A}_f) \otimes \pi_0(K_{n,\infty})$. Similarly, $\vartheta_{\Sigma,\eta} \in H^{b_{n-1}}_{\mathrm{cusp}}(S_{n-1}(\iota^*K_f), \mathcal{M}^\vee_\lambda)(\tilde{\Sigma}_f)$.

We recall that cuspidal cohomology injects into cohomology with compact supports, i.e., $H^*_{\mathrm{cusp}} \hookrightarrow H^*_c$. (See [6, p.129].) Hence $\vartheta_{\Pi,\epsilon}$ is a class in $H^{b_n}_c(S_n(K_f), \mathcal{M}^\vee_\mu)$, and similarly, $\vartheta_{\Sigma,\eta}$ lies in $H^{b_{n-1}}_c(S_{n-1}(\iota^*K_f), \mathcal{M}^\vee_\lambda)$. (In this context, it is also helpful to bear in mind that cuspidal cohomology in fact injects into interior cohomology $H^*_! := \mathrm{Image}(H^*_c \to H^*)$. This is useful especially when dealing with rational structures; see [6, Proof of Théorème 3.19] or our paper [33, §3.3].)

We remind the reader that the map $\iota : S_{n-1}(\iota^*K_f) \to S_n(K_f)$ is a proper map. Consider the pull back $\iota^*\vartheta_{\Pi,\epsilon}$ of $\vartheta_{\Pi,\epsilon}$ via ι, which gives us a class in $H^{b_n}_c(S_{n-1}(\iota^*K_f), \iota^*\mathcal{M}^\vee_\mu)$, where $\iota^*\mathcal{M}^\vee_\mu$ is the sheaf on $S_{n-1}(\iota^*K_f)$ attached to the restriction to G_{n-1} of the representation $M^\vee_\mu$. We now define a certain pairing $\langle \vartheta_{\Sigma,\eta}, \iota^*\vartheta_{\Pi,\epsilon}\rangle_{\mathcal{C}(\iota^*K_f)}$, toward which we recall the construction of a cycle $\mathcal{C}(\iota^*K_f)$.

3.2.3. *The Harder-Mahnkopf cycle.* We first explain the general principle of the construction. Let M be a smooth connected orientable manifold of dimension d, $\overline{M}$ a compactification of M, and $\partial\overline{M}$ the boundary of $\overline{M}$. Suppose that $M = \overline{M} - \partial\overline{M} = \mathrm{int}(\overline{M})$, and that M and $\overline{M}$ have the same homotopy type. (We should keep in mind the Borel-Serre compactification of a locally symmetric space.) We have the following isomorphisms based on Poincaré duality:

$$\begin{aligned} \mathrm{Hom}(H^d_c(M,\mathbb{Z}),\mathbb{Z}) &\simeq \mathrm{Hom}(H_0(M,\mathbb{Z}),\mathbb{Z}) \simeq H^0(\overline{M},\mathbb{Z}) \\ &\simeq H_d(\overline{M},\partial\overline{M},\mathbb{Z}) \simeq \mathbb{Z} =: \langle[\vartheta_M]\rangle. \end{aligned}$$

To talk about $H^d_c(M,\mathbb{Z})$ we have transported the $\mathbb{Z}$-structure on singular cohomology via the de Rham isomorphism. The fundamental class $[\vartheta_M]$ is well-defined up to a sign, and by the above isomorphisms, induces a functional $H^d_c(M,\mathbb{Z}) \to \mathbb{Z}$ which is nothing but integrating a compactly supported differential form of degree d over the entire manifold M (with the chosen orientation, i.e., the choice of $[\vartheta_M]$). If the manifold M is disconnected, but has finitely many connected components, then in certain situations including the one we are interested in, it makes sense to choose the fundamental classes for each connected component in a *consistent* manner.

We digress a little to note that the above construction has good rationality properties. We recall ([33, §3]) that by definition of the action of $\sigma \in \mathrm{Aut}(\mathbb{C})$ on de Rham cohomology, as well as on cohomology with compact supports, one has

$$\sigma\left(\int_M \omega\right) = \int_M \omega^\sigma$$

for any $\omega \in H^d_c(M,\mathbb{C})$, which may be re-written as $\sigma(\langle[\vartheta_M],\omega\rangle) = \langle[\vartheta_M],\omega^\sigma\rangle$.

We now briefly review the Harder-Mahnkopf cycle, while referring the reader to [29, 5.1.1] for all finer details. Recall that $K^1_{n,\infty} := \mathrm{SO}(n) < G^0_{n,\infty} = G_n(\mathbb{R})^0$. For any open compact subgroup K_f of $G_n(\mathbb{A}_f)$ consider the manifold

$$F_n(K_f) = G_n(\mathbb{Q})\backslash G_n(\mathbb{A})/K^1_{n,\infty}K_f.$$

We let $d_n = n(n+1)/2 = \dim(F_n(K_f))$. The connected components $F_{n,x}(K_f)$ of the manifold $F_n(K_f)$ are parametrized by $x \in \mathbb{Q}^*\backslash\mathbb{A}^\times/\mathbb{R}_{>0}\det(K_f)$; indeed, for any such x, let $g_x \in G_n(\mathbb{A}_f)$ be such that $\det(g_x) = x$, then $F_{n,x}(K_f)$ is identified with $\Gamma_x\backslash G_n(\mathbb{R})^0/K^1_{n,\infty}$ for the discrete subgroup $\Gamma_x = G_n(\mathbb{Q}) \cap g_xK_fg_x^{-1}$. These notations also apply to G_{n-1} with any open compact subgroup R_f of $G_{n-1}(\mathbb{A}_f)$. Choose an orientation on $X_{n-1} := G_{n-1}(\mathbb{R})^0/K^1_{n-1,\infty}$. Via the canonical map $X_{n-1} \to \Gamma_x\backslash X_{n-1} = F_{n-1,x}(K_f)$ we get a fundamental class $[\vartheta_{x,R_f}]$ on $F_{n-1,x}(R_f)$, i.e., $[\vartheta_{x,R_f}] \in H_{d_{n-1}}(\overline{F}_{n-1,x}(R_f),\partial\overline{F}_{n-1,x}(R_f),\mathbb{Z})$.

At this point, it is convenient to work with $\mathbb{Q}$-coefficients. (Indeed, ultimately, it suffices to work with the ring obtained by inverting a finite set of primes determined by the primes where Π and Σ are ramified.) Now define

$$\mathcal{C}(R_f) = \frac{1}{\mathrm{vol}(R_f)} \sum_{x\in\mathbb{Q}^*\backslash\mathbb{A}^\times/\mathbb{R}_{>0}\det(R_f)} [\vartheta_{x,R_f}]$$

which is the required cycle in $H_{d_{n-1}}(\overline{F}_{n-1}(R_f),\partial\overline{F}_{n-1}(R_f),\mathbb{Q})$.

Recall that $\pi_0(G_n)$ (resp., $\pi_0(K_{n,\infty})$) is the group of connected components of $G_n(\mathbb{R})$ (resp., $\mathrm{O}(n)Z_n(\mathbb{R})$). We identify $\pi_0(K_{n,\infty}) \simeq \pi_0(G_n) \simeq \mathbb{Z}/2$. The nontrivial element may be taken to be represented by $\delta_n = \mathrm{diag}(-1,1,\dots,1)$. Right translations by δ_n on $G_n(\mathbb{A})$, denoted r_{δ_n}, induces an action of π_0 on $F_n(K_f)$, and by functoriality induces an action, denoted $r^*_{\delta_n}$, on its (co-)homology groups. Applying these considerations to G_{n-1}, we get an action of $\pi_0(G_{n-1})$ on the cycle $\mathcal{C}(R_f)$ which is described in the following

Lemma 3.4. *For any open compact subgroup R_f of $G_{n-1}(\mathbb{A}_f)$, the action of δ_{n-1} on the cycle $\mathcal{C}(R_f)$ is given by:*

$$r^*_{\delta_{n-1}}\mathcal{C}(R_f) = (-1)^n\mathcal{C}(R_f).$$

Proof. See Lemma 5.1.3 and the table in 5.2.2 of Mahnkopf [29]. (This is a generalization of the fact that δ_2 switches the (orientations on the) upper and lower half planes.) □

3.2.4. *The pairing $\langle\vartheta_{\Sigma,\eta},\vartheta_{\Pi,\epsilon}\rangle_{\mathcal{C}(R_f)}$.* Now assume that K_f is an open compact subgroup of $G_n(\mathbb{A}_f)$, which for convenience may be taken to be a principal congruence subgroup of $G_n(\widehat{\mathbb{Z}})$. We let $R_f := \iota^*K_f$ which is an open compact subgroup of $G_{n-1}(\widehat{\mathbb{Z}})$. The map ι induces a proper map $\iota : F_{n-1}(R_f) \to S_n(K_f)$, which in turn induces a mapping

$$\iota^* : H^\bullet_c(S_n(K_f),\mathcal{M}^\vee_\mu) \to H^\bullet_c(F_{n-1}(R_f),\iota^*\mathcal{M}^\vee_\mu).$$

Also, the canonical map $p : F_{n-1}(R_f) \to S_{n-1}(R_f)$ induces a mapping

$$p^* : H_c^\bullet(S_{n-1}(R_f), \mathcal{M}_\lambda^\vee) \to H_c^\bullet(F_{n-1}(R_f), \mathcal{M}_\lambda^\vee).$$

We invoke the hypothesis $\mu^\vee \succ \lambda$ which implies that M_λ appears in $M_\mu^\vee|_{G_{n-1}} = \iota^* M_\mu^\vee$, and, in fact, it appears with multiplicity one. We fix a G_{n-1}-equivariant pairing, defined over $\mathbb{Q}$, and unique up to $\mathbb{Q}^*$, which we write as

$$\langle \cdot, \cdot \rangle : M_\lambda^\vee \times \iota^* M_\mu^\vee \to \mathbb{Q} \tag{3.5}$$

and denote the corresponding morphism of sheaves as $\langle \cdot, \cdot \rangle : \mathcal{M}_\lambda^\vee \otimes \iota^* \mathcal{M}_\mu^\vee \to \underline{\mathbb{Q}}$. Cup product together with the above pairing gives a map:

$$\langle \cdot, \cdot \rangle \circ \cup : H_c^{b_{n-1}}(F_{n-1}(R_f), \mathcal{M}_\lambda^\vee) \times H_c^{b_n}(F_{n-1}(R_f), \iota^* \mathcal{M}_\mu^\vee) \to H_c^{d_{n-1}}(F_{n-1}(R_f), \underline{\mathbb{Q}}).$$

This makes sense since $b_{n-1} + b_n = d_{n-1}$. We will abbreviate this map simply by $\cup$. We digress for a moment to remind the reader that cupping cohomology classes, which makes sense in the context of singular cohomology, is the same as wedging cohomology classes, which makes sense in the context of de Rham cohomology. (See Griffiths-Harris [14].) To control rationality properties, it is best to think of the cup product, but to actually compute the pairing–as we will do later–it is best to think in terms of the wedge product. This also permits us to write

$$p^* \vartheta_{\Sigma,\eta} \cup \iota^* \vartheta_{\Pi,\epsilon} = p^* \vartheta_{\Sigma,\eta} \wedge \iota^* \vartheta_{\Pi,\epsilon}.$$

We now define the required pairing as

$$\langle \vartheta_{\Sigma,\eta}, \vartheta_{\Pi,\epsilon} \rangle_{\mathcal{C}(R_f)} := \langle \mathcal{C}(R_f), p^* \vartheta_{\Sigma,\eta} \cup \iota^* \vartheta_{\Pi,\epsilon} \rangle = \int_{\mathcal{C}(R_f)} p^* \vartheta_{\Sigma,\eta} \wedge \iota^* \vartheta_{\Pi,\epsilon} \tag{3.6}$$

where the second equality is given by Poincaré duality as described in 3.2.3. (See also [29, Diagram (5.3)].)

3.2.5. *The pairing at infinity and a nonvanishing hypothesis.* We recall again that the class $\vartheta_{\Pi,\epsilon}$ is the image of a certain global finite Whittaker vector w_{Π_f} under the map $\mathcal{F}_{\Pi_f,\epsilon,[\Pi_\infty]}$. (All these comments also apply to $\vartheta_{\Sigma,\eta}$.) We recall [33, §3.3] that this map is the composition of the three isomorphisms:

$$\begin{aligned} W(\Pi_f) &\longrightarrow W(\Pi_f) \otimes H^{b_n}(\mathfrak{g}_{n,\infty}, K_{n,\infty}^0; W(\Pi_\infty) \otimes M_\mu^\vee)(\epsilon) \\ &\longrightarrow H^{b_n}(\mathfrak{g}_{n,\infty}, K_{n,\infty}^0; W(\Pi) \otimes M_\mu^\vee)(\epsilon) \\ &\longrightarrow H^{b_n}(\mathfrak{g}_{n,\infty}, K_{n,\infty}^0; V_\Pi \otimes M_\mu^\vee)(\epsilon), \end{aligned}$$

where the first map is $w_f \mapsto w_f \otimes [\Pi_\infty]$; the second map is the obvious one; and the third map is the map induced in cohomology by the inverse of the map which gives the Fourier coefficient of a cusp form in V_Π–the space of functions in $\mathcal{A}_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ which realizes Π. In particular, in computing the pairing $\langle \vartheta_{\Sigma,\eta}, \iota^* \vartheta_{\Pi,\epsilon} \rangle_{\mathcal{C}(R_f)}$, we will be computing a pairing at infinity, and a pairing with the finite vectors w_{Π_f} and w_{Σ_f}. The latter is indeed the Rankin–Selberg integral (at $s = 1/2$) appearing in the left hand side of Proposition 3.1. We now discuss the pairing at infinity.

To compute the pairing at infinity, we follow the argument in [29, §5.1.4]. Fix a basis $\{\mathbf{x}_i\}$ for $(\mathfrak{g}_{n,\infty}/\mathfrak{k}_{n,\infty})^*$, and a basis $\{\mathbf{y}_j\}$ for $(\mathfrak{g}_{n-1,\infty}/\mathfrak{k}_{n-1,\infty})^*$, such that $\iota^* \mathbf{x}_j = \mathbf{y}_j$ for all $1 \le j \le \dim(\mathfrak{g}_{n-1,\infty}/\mathfrak{k}_{n-1,\infty})^* = \dim(X_{n-1}) = d_{n-1}$, and $\iota^* \mathbf{x}_i = 0$ if $i > d_{n-1}$. We further

note that $\mathbf{y}_1 \wedge \mathbf{y}_2 \wedge \cdots \wedge \mathbf{y}_{d_{n-1}}$ corresponds to a $G_{n-1}(\mathbb{R})^0$-invariant measure on X_{n-1}. Let $\{m_\alpha\}$ (resp., $\{m_\beta\}$) be a $\mathbb{Q}$-basis for $M_\mu^\vee$ (resp., $M_\lambda^\vee$), and recall that we have a pairing $\langle \cdot, \cdot \rangle$ between these modules as in (3.5). Now the class $[\Pi_\infty]$ is represented by a $K_{n,\infty}^0$-invariant element in $\wedge^{b_n}(\mathfrak{g}_{n,\infty}/\mathfrak{k}_{n,\infty})^* \otimes W(\Pi_\infty) \otimes M_\mu^\vee$ which we write as

$$[\Pi_\infty] = \sum_{\mathbf{i}=i_1<\cdots<i_{b_n}} \sum_\alpha \mathbf{x}_\mathbf{i} \otimes w_{\infty,\mathbf{i},\alpha} \otimes m_\alpha, \tag{3.7}$$

where $w_{\infty,\mathbf{i},\alpha} \in W(\Pi_\infty, \psi_\infty)$, and similarly, $[\Sigma_\infty]$ is represented by a $K_{n-1,\infty}^0$-invariant element in $\wedge^{b_{n-1}}(\mathfrak{g}_{n-1,\infty}/\mathfrak{k}_{n-1,\infty})^* \otimes W(\Sigma_\infty) \otimes M_\lambda^\vee$ which we write as:

$$[\Sigma_\infty] = \sum_{\mathbf{j}=j_1<\cdots<j_{b_{n-1}}} \sum_\beta \mathbf{y}_\mathbf{j} \otimes w_{\infty,\mathbf{j},\beta} \otimes m_\beta, \tag{3.8}$$

with $w_{\infty,\mathbf{j},\beta} \in W(\Sigma_\infty, \psi_\infty^{-1})$. We now define a pairing at infinity by

$$\langle [\Pi_\infty], [\Sigma_\infty] \rangle = \sum_{\mathbf{i},\mathbf{j}} s(\mathbf{i},\mathbf{j}) \sum_{\alpha,\beta} \langle m_\beta, m_\alpha \rangle \Psi_\infty(1/2, w_{\infty,\mathbf{i},\alpha}, w_{\infty,\mathbf{j},\beta}) \tag{3.9}$$

where $s(\mathbf{i},\mathbf{j}) \in \{0,-1,1\}$ is defined by $\iota^*\mathbf{x}_\mathbf{i} \wedge \mathbf{y}_\mathbf{j} = s(\mathbf{i},\mathbf{j})\mathbf{y}_1 \wedge \mathbf{y}_2 \wedge \cdots \wedge \mathbf{y}_{d_{n-1}}$. Recall that $\Psi_\infty(1/2, w_{\infty,\mathbf{i},\alpha}, w_{\infty,\mathbf{j},\beta})$ is defined only after meromorphic continuation; see Cogdell–Piatetskii-Shapiro [8, Theorem 1.2]. Note that the assumption '$s = 1/2$ is critical' ensures that the integrals $\Psi_\infty(1/2, w_{\infty,\mathbf{i},\alpha}, w_{\infty,\mathbf{j},\beta})$ are all finite, hence $\langle [\Pi_\infty], [\Sigma_\infty] \rangle$ is finite. We now make the following nonvanishing hypothesis about this pairing at infinity:

Hypothesis 3.10. $\langle [\Pi_\infty], [\Sigma_\infty] \rangle \neq 0$.

This nonvanishing hypothesis is currently a limitation of this technique. It has shown up in several other works based on the same, or at any rate similar, techniques. See for instance Ash-Ginzburg [1], Harris [16], Kasten-Schmidt [22], Kazhdan-Mazur-Schmidt [23], Mahnkopf [29], and Schmidt [36]. It is widely hoped that this assumption is valid; for example, Mahnkopf [29, §6] proves a necessary condition for this nonvanishing assumption, Schmidt [36] proved it for $n = 3$ in the case of trivial coefficients ($\mu = 0$ and $\lambda = 0$), and Kasten-Schmidt [22, §4] have recently proved it for $n = 3$ for nontrivial coefficients. *It is an important technical problem to be able to prove this nonvanishing hypothesis.*

For the rest of this paper we assume that Hypothesis 3.10 is valid. Observe that the quantity $\langle [\Pi_\infty], [\Sigma_\infty] \rangle$ depends only on the weights μ and λ, since the weight μ determines the infinitesimal character of $M_\mu^\vee$ which in turn determines Π_∞ ([33, §5.1]), and similarly, since λ determines Σ_∞. We now define, what may loosely be called as the period at infinity, a nonzero complex number $p_\infty(\mu, \lambda)$ given by:

$$p_\infty(\mu, \lambda) := \frac{1}{\langle [\Pi_\infty], [\Sigma_\infty] \rangle}. \tag{3.11}$$

Ultimately, if one is able to explicitly compute everything, then one should expect $p_\infty(\mu, \lambda)$ to be a power of $(2\pi i)$.

3.2.6. *The main identity for the central critical value of Rankin–Selberg L-functions.*

Theorem 3.12 (Main Identity)**.** *Let* Π *be a regular algebraic cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$*, and let* Σ *be a regular algebraic cuspidal automorphic representation of* $\mathrm{GL}_{n-1}(\mathbb{A}_\mathbb{Q})$*. Let* $\mu \in X_0^+(T_n)$ *be such that* $\Pi \in \mathrm{Coh}(G_n, \mu^\vee)$*, and let* $\lambda \in X_0^+(T_{n-1})$ *be such that* $\Sigma \in \mathrm{Coh}(G_{n-1}, \lambda^\vee)$*. Assume that* $\mu^\vee \succ \lambda$*, and that* $s = 1/2$ *is critical for* $L_f(s, \Pi \times \Sigma)$*. We attach a canonical pair of signs* $\epsilon, \eta \in \{\pm\}$ *to the pair* (Π, Σ) *as follows:*

(1) $\epsilon = (-1)^n \eta$.

(2)
- *If* n *is odd then let* $\epsilon = \omega_{\Pi_\infty}(-1)(-1)^{\mathrm{wt}(\mu)/2}$;
- *if* n *is even then let* $\eta = \omega_{\Sigma_\infty}(-1)(-1)^{\mathrm{wt}(\lambda)/2}$.

Let w_{Π_f} *and* w_{Σ_f} *be the Whittaker vectors defined in 3.1.4. We let* K_f *be any open compact subgroup of* $G_n(\mathbb{A}_f)$ *which fixes* w_{Π_f} *and such that* $R_f := \iota^* K_f$ *fixes* w_{Σ_f}*. We let* $\vartheta^0_{\Pi,\epsilon}$ *and* $\vartheta^0_{\Sigma,\eta}$ *be the normalized classes defined in (3.2) and (3.3). There exists nonzero complex numbers* $p^\epsilon(\Pi)$ *and* $p^\eta(\Sigma)$ *as in 3.2.1, and assuming the validity of Hypothesis 3.10 there is a nonzero complex number* $p_\infty(\mu, \lambda)$ *as in 3.2.5 such that*

$$\frac{L_f(1/2, \Pi \times \Sigma)}{p^\epsilon(\Pi)\, p^\eta(\Sigma)\, p_\infty(\mu, \lambda)} \;=\; \frac{\prod_{v \in S_\Sigma} L(1/2, \Pi_v \times \Sigma_v)}{\mathrm{vol}(\Sigma) \prod_{v \notin S_\Sigma \cup \{\infty\}} c_{\Pi_v}} \langle \vartheta^0_{\Sigma,\eta}, \vartheta^0_{\Pi,\epsilon} \rangle_{\mathcal{C}(R_f)},$$

where the pairing on the right hand side is defined in (3.6), the nonzero rational number $\mathrm{vol}(\Sigma)$ *is as in Proposition 3.1, and* c_{Π_v} *is defined in 3.1.3.*

Proof. By definition of the normalization of the cohomology classes, it suffices to prove

$$\frac{L_f(1/2, \Pi \times \Sigma)}{p_\infty(\mu, \lambda)} \;=\; \frac{\prod_{v \in S_\Sigma} L(1/2, \Pi_v \times \Sigma_v)}{\mathrm{vol}(\Sigma) \prod_{v \notin S_\Sigma \cup \{\infty\}} c_{\Pi_v}} \langle \vartheta_{\Sigma,\eta}, \vartheta_{\Pi,\epsilon} \rangle_{\mathcal{C}(R_f)}.$$

By definition of the pairing at infinity, it suffices then to verify

$$\langle \vartheta_{\Sigma,\eta}, \vartheta_{\Pi,\epsilon} \rangle_{\mathcal{C}(R_f)} = \frac{\mathrm{vol}(\Sigma) \prod_{v \notin S_\Sigma \cup \{\infty\}} c_{\Pi_v}}{\prod_{v \in S_\Sigma} L(1/2, \Pi_v \times \Sigma_v)} \langle [\Pi_\infty], [\Sigma_\infty] \rangle \, L_f(1/2, \Pi \times \Sigma). \tag{3.13}$$

The class $\vartheta_{\Pi,\epsilon} \in H^{b_n}(\mathfrak{g}_{n,\infty}, K^0_{n,\infty}; V_\Pi \otimes M^\vee_\mu)(\epsilon)$, as in 3.2.5, is represented by a $K^0_{n,\infty}$-invariant element in $\wedge^{b_n}(\mathfrak{g}_{n,\infty}/\mathfrak{k}_{n,\infty})^* \otimes V_\Pi \otimes M^\vee_\mu$ which we write as

$$\vartheta_{\Pi,\epsilon} = \sum_{\mathbf{i} = i_1 < \cdots < i_{b_n}} \sum_\alpha \mathbf{x_i} \otimes \phi_{\mathbf{i},\alpha} \otimes m_\alpha.$$

Similarly, we write $\vartheta_{\Sigma,\eta}$ as

$$\vartheta_{\Sigma,\eta} = \sum_{\mathbf{j} = j_1 < \cdots < j_{b_{n-1}}} \sum_\beta \mathbf{y_j} \otimes \phi_{\mathbf{j},\beta} \otimes m_\beta,$$

representing a $K^0_{n-1,\infty}$-invariant element in $\wedge^{b_{n-1}}(\mathfrak{g}_{n-1,\infty}/\mathfrak{k}_{n-1,\infty})^* \otimes V_\Sigma \otimes M^\vee_\lambda$. Let $w_{\mathbf{i},\alpha}$ be the Whittaker vector in $W(\Pi, \psi)$ corresponding to $\phi_{\mathbf{i},\alpha}$, and similarly, $w_{\mathbf{j},\beta}$ be the Whittaker vector in $W(\Sigma, \psi^{-1})$ corresponding to $\phi_{\mathbf{j},\beta}$. Unravelling the definitions, we have the decompositions

$$w_{\mathbf{i},\alpha} = w_{\infty,\mathbf{i},\alpha} \otimes w_{\Pi_f}, \quad \text{and} \quad w_{\mathbf{j},\beta} = w_{\infty,\mathbf{j},\beta} \otimes w_{\Sigma_f},$$

where the vectors at infinity are exactly as in (3.7) and (3.8).

To verify (3.13) we begin with the definition of the pairing

$$\langle \vartheta_{\Sigma,\eta}, \vartheta_{\Pi,\epsilon}\rangle_{\mathcal{C}(R_f)} = \int_{\mathcal{C}(R_f)} p^*\vartheta_{\Sigma,\eta} \wedge \iota^*\vartheta_{\Pi,\epsilon},$$

and observe that the integral on the right hand side is stable under the action of π_0 exactly when $\epsilon\eta = (-1)^n$; this may be seen by using Lemma 3.4 (just as in [29, 5.2.2]). Next, we note that the right hand side may be written as

$$\frac{1}{\mathrm{vol}(R_f)} \sum_{\mathbf{i},\mathbf{j},\alpha,\beta} s(\mathbf{i},\mathbf{j})\langle m_\beta, m_\alpha\rangle \int_{G_{n-1}(\mathbb{Q})\backslash G_{n-1}(\mathbb{A})/K^1_{n-1,\infty}R_f} \phi_{\mathbf{i},\alpha}(\iota(g))\phi_{\mathbf{j},\beta}(g)\,dg.$$

By the choice of the measure dg, this simplifies to

$$\sum_{\mathbf{i},\mathbf{j},\alpha,\beta} s(\mathbf{i},\mathbf{j})\langle m_\beta, m_\alpha\rangle \int_{G_{n-1}(\mathbb{Q})\backslash G_{n-1}(\mathbb{A})} \phi_{\mathbf{i},\alpha}(\iota(g))\phi_{\mathbf{j},\beta}(g)\,dg.$$

The inner integral is nothing but $I(1/2, \phi_{\mathbf{i},\alpha}, \phi_{\mathbf{j},\beta})$. Applying Proposition 3.1 we get

$$\frac{\mathrm{vol}(\Sigma)\prod_{v\notin S_\Sigma\cup\{\infty\}} c_{\Pi_v}}{\prod_{v\in S_\Sigma} L(1/2, \Pi_v \times \Sigma_v)} L_f(1/2, \Pi\times\Sigma) \sum_{\mathbf{i},\mathbf{j},\alpha,\beta} s(\mathbf{i},\mathbf{j})\langle m_\beta, m_\alpha\rangle \Psi_\infty(1/2, w_{\infty,\mathbf{i},\alpha}, w_{\infty,\mathbf{j},\beta}).$$

The proof follows from the definition of $\langle[\Pi_\infty], [\Sigma_\infty]\rangle$. □

3.3. **Proof of Theorem 1.1.** The proof follows by applying $\sigma \in \mathrm{Aut}(\mathbb{C})$ to the main identity in Theorem 3.12. We now would like to know the Galois equivariance of all the quantities on the right hand side of the main identity. This we delineate in the following propositions:

Proposition 3.14. *Let $\varpi \in H_c^{b_n}(S_n(K_f), \mathcal{M}_{\mu^\vee})$, and $\varsigma \in H_c^{b_{n-1}}(S_{n-1}(\iota^*K_f), \mathcal{M}_{\lambda^\vee})$. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have*

$$\sigma\left(\langle \varsigma\,,\,\varpi\rangle_{\mathcal{C}(R_f)}\right) = \langle {}^\sigma\varsigma\,,\,{}^\sigma\varpi\rangle_{\mathcal{C}(R_f)}.$$

Proof. This follows from the well-known Galois equivariance property of Poincaré duality (see, for example, Mahnkopf [28, proof of Lemma 1.2]), coupled with the fact that the maps ι^* and p^* are Galois equivariant. □

Proposition 3.15. *The classes $\vartheta^0_{\Pi,\epsilon}$ and $\vartheta^0_{\Sigma,\eta}$, constructed in 3.2.2, have the following behaviour under $\sigma \in \mathrm{Aut}(\mathbb{C})$:*

$${}^\sigma\vartheta^0_{\Pi,\epsilon} = \sigma(\omega_{\Sigma_f}(t_\sigma))\vartheta^0_{\Pi^\sigma,\epsilon}, \quad {}^\sigma\vartheta^0_{\Sigma,\eta} = \vartheta^0_{\Sigma^\sigma,\eta}.$$

Proof. By definition of the classes, and the Galois equivariance of $\mathcal{F}^0$, we have

$${}^\sigma\vartheta^0_{\Pi,\epsilon} = {}^\sigma\mathcal{F}^0_{\Pi_f,\epsilon,[\Pi_\infty]}(w_{\Pi_f}) = \mathcal{F}^0_{\Pi^\sigma_f,\epsilon,[\Pi_\infty]}({}^\sigma w_{\Pi_f}).$$

Next, we note that by the choice of the vector w_{Π_f}, we have

$${}^\sigma w_{\Pi_f} = {}^\sigma(\otimes_{v\notin S_\Sigma} w^0_{\Pi_v} \otimes_{v\in S_\Sigma} w_{\Pi,v}) = \otimes_{v\notin S_\Sigma} {}^\sigma w^0_{\Pi_v} \otimes_{v\in S_\Sigma} {}^\sigma w_{\Pi,v},$$

the second equality is due to the compatibility of local and global actions of σ . For $v \notin S_\Sigma$ we know that ${}^\sigma w^0_{\Pi_v} = w^0_{\Pi^\sigma_v}$. However, for $v \in S_\Sigma$, we note first that the support of ${}^\sigma w_{\Pi,v}$

restricted to G_{n-1} is also the same double coset $N_{n-1}(\mathbb{Q}_v)t_{\Sigma_v}K_{n-1}(\mathfrak{f}(\Sigma_v))$, and on this double coset it is given by

$$\begin{aligned} {}^{\sigma}w_{\Pi,v}|_{G_{n-1}}(ut_{\Sigma_v}k) &= \sigma\left(w_{\Pi,v}|_{G_{n-1}}\left(\left(\begin{pmatrix} t_{\sigma,v}^{-(n-1)} & & & \\ & t_{\sigma,v}^{-(n-2)} & & \\ & & \dots & \\ & & & t_{\sigma,v}^{-1}\end{pmatrix} ut_{\Sigma_v}k\right)\right)\right) \\ &= \sigma(\psi_v(t_\sigma^{-1}u)\omega_{\Sigma_v}^{-1}(t_{\sigma,v}^{-1}k_{n-1,n-1})) \\ &= \sigma(\omega_{\Sigma_v}(t_{\sigma,v}))w_{\Pi^\sigma,v}|_{G_{n-1}}(ut_{\Sigma_v}k). \end{aligned}$$

Hence, for $v \in S_\Sigma$ we have ${}^{\sigma}w_{\Pi,v} = \sigma(\omega_{\Sigma_v}(t_{\sigma,v}))w_{\Pi_v^\sigma}$. Noting that $\otimes_{v\in S_\Sigma}\omega_{\Sigma_v}(t_{\sigma,v}) = \omega_{\Sigma_f}(t_\sigma)$, we get ${}^{\sigma}w_{\Pi_f} = \sigma(\omega_{\Sigma_f}(t_\sigma))w_{\Pi_f^\sigma}$, which finishes the proof of the first assertion of the proposition. The proof of Galois equivariance of the class $\vartheta^0_{\Sigma,\eta}$ is similar (and simpler). $\square$

For later use we note that the above variance for $\vartheta^0_{\Pi,\epsilon}$ may also be stated in terms of Gauss sums.

Corollary 3.16. *For any $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have*

$${}^{\sigma}\vartheta^0_{\Pi,\epsilon} = \frac{\sigma(\mathcal{G}(\omega_{\Sigma_f}))}{\mathcal{G}(\omega_{\Sigma_f^\sigma})}\vartheta^0_{\Pi^\sigma,\epsilon}$$

Proof. Follows from the above proposition and (2.1). $\square$

Proposition 3.17. *For a finite place v of $\mathbb{Q}$, let Π_v and Σ_v be (any) irreducible admissible representations of $G_n(\mathbb{Q}_v)$ and $G_{n-1}(\mathbb{Q}_v)$. Then*

$$\sigma(L(1/2, \Pi_v \times \Sigma_v)) = L(1/2, \Pi_v^\sigma \times \Sigma_v^\sigma).$$

Proof. Let $F = \mathbb{Q}_v$, or for that matter, any non-archimedean local field with its associated baggage of notations like $\mathcal{O}$, $\mathcal{P}$, q, etc. Let π be any irreducible admissible representation of $\mathrm{GL}_m(F)$. From Clozel [6, Lemma 4.6] we have

$${}^{\sigma}L\left(s + \frac{1-m}{2}, \pi\right) = L\left(s + \frac{1-m}{2}, \pi^\sigma\right).$$

In the left hand side, if $L(s+(1-m)/2, \pi) = P(q^{-s})^{-1}$ for a polynomial $P(X) \in \mathbb{C}[X]$ with $P(0) = 1$, then ${}^{\sigma}P(q^{-s})$ is obtained by applying σ to the coefficients of $P(X)$. Now assume that m is even. Then

$$\begin{aligned} \sigma(L(1/2,\pi)) &= \sigma\left(L\left(s + \frac{1-m}{2}, \pi\right)|_{s=m/2}\right) \\ &= \sigma(P(q^{-m/2})^{-1}) \\ &= {}^{\sigma}P(q^{-m/2})^{-1} \quad \text{(since } m \text{ is even)} \\ &= {}^{\sigma}L(1/2,\pi). \end{aligned}$$

From the above mentioned lemma we have

$$\sigma(L(1/2,\pi)) = L(1/2,\pi^\sigma). \tag{3.18}$$

We need a result of Henniart about the local Langlands correspondence for $\mathrm{GL}_m(F)$. We denote this correspondence as $\pi \mapsto \tau(\pi)$ and $\tau \mapsto \pi(\tau)$ between irreducible admissible representations π of $\mathrm{GL}_m(F)$ and m-dimensional semisimple representations τ of the Weil-Deligne group $W'_F = W_F \times \mathrm{SL}_2(\mathbb{C})$. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$, we let ϵ_σ denote the quadratic character $x \mapsto \sigma(|x|^{1/2})/|x|^{1/2}$ of F^*. From Henniart [17, 7.4] we have

$$\pi(\tau)^\sigma = \pi(\tau^\sigma)\epsilon_\sigma^{m-1}, \text{ and } \tau(\pi)^\sigma = \tau(\pi^\sigma)\epsilon_\sigma^{m-1}. \tag{3.19}$$

In [17] the Langlands correspondence is stated between the Grothendieck group generated by irreducible representations of the Weil group W_F on the one hand and the Grothendieck group generated by irreducible supercuspidal representations on the other. In particular, (3.19) is stated for such representations. However, one can easily see that (3.19) remains true as we have stated it if one defines the action of $\sigma \in \mathrm{Aut}(\mathbb{C})$ on semisimple representations of W'_F by

$$(\tau_1 \otimes \mathcal{S}_{m_1} \oplus \cdots \oplus \tau_r \otimes \mathcal{S}_{m_r})^\sigma = \tau_1^\sigma \otimes \mathcal{S}_{m_1} \oplus \cdots \oplus \tau_1^\sigma \otimes \mathcal{S}_{m_1}$$

for irreducible representations τ_i of W_F, and integers m_i, where for any integer $k \geq 1$ the k-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ is denoted $\mathcal{S}_k$.

Now, let π_1 and π_2 be irreducible admissible representations of $\mathrm{GL}_{m_1}(F)$ and $\mathrm{GL}_{m_2}(F)$, respectively. Define the 'automorphic tensor product' by $\pi_1 \boxtimes \pi_2 := \pi(\tau(\pi_1) \otimes \tau(\pi_2))$. One can check from (3.19) that for any $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have

$$(\pi_1 \boxtimes \pi_2)^\sigma = (\pi_1^\sigma \boxtimes \pi_2^\sigma) \otimes \epsilon_\sigma^{(1-m_1)(1-m_2)}. \tag{3.20}$$

The proposition follows from (3.18) and (3.20) by taking $m_1 = n$, $m_2 = n-1$, $\pi_1 = \Pi_v$, $\pi_2 = \Sigma_v$, and $\pi = \pi_1 \boxtimes \pi_2$, while keeping in mind that $L(s, \pi_1 \times \pi_2) = L(s, \pi_1 \boxtimes \pi_2)$. □

Albeit the above proposition is not hard to prove, we wish to emphasize the fact that it is a crucial ingredient in our paper. The moral being that the possibly transcendental parts of special values of L-functions are already captured by partial L-functions, i.e., we can ignore finitely many places as these local L-values are in the rationality field.

Proposition 3.21.

$$\sigma(c_{\Pi_v}) = c_{\Pi_v^\sigma}.$$

Proof. See Mahnkopf [29, p.621], where it is mentioned that the proof is the same argument as in the proof of [28, Proposition 2.3(c)]. □

Proof of Theorem 1.1. Apply $\sigma \in \mathrm{Aut}(\mathbb{C})$ to the main identity in Theorem 3.12 to get

$$\sigma\left(\frac{L_f(1/2, \Pi \times \Sigma)}{p^\epsilon(\Pi)p^\eta(\Sigma)p_\infty(\mu,\lambda)}\right) = \sigma\left(\frac{\prod_{v \in S_\Sigma} L(1/2, \Pi_v \times \Sigma_v)}{\mathrm{vol}(\Sigma)\prod_{v \notin S_\Sigma} c_{\Pi_v}} \langle \vartheta^0_{\Sigma,\eta}, \vartheta^0_{\Pi,\epsilon}\rangle_{\mathcal{C}(R_f)}\right).$$

Applying Propositions 3.14, 3.17, 3.21, and Corollary 3.16 to the right hand side we have

$$\frac{\prod_{v \in S_{\Sigma^\sigma}} L(1/2, \Pi_v^\sigma \times \Sigma_v^\sigma)}{\mathrm{vol}(\Sigma^\sigma)\prod_{v \notin S_{\Sigma^\sigma}} c_{\Pi_v^\sigma}} \frac{\sigma(\mathcal{G}(\omega_{\Sigma_f}))}{\mathcal{G}(\omega_{\Sigma_f^\sigma})} \langle \vartheta^0_{\Sigma^\sigma,\eta}, \vartheta^0_{\Pi^\sigma,\epsilon}\rangle_{\mathcal{C}(R_f)} = \frac{\sigma(\mathcal{G}(\omega_{\Sigma_f}))}{\mathcal{G}(\omega_{\Sigma_f^\sigma})} \frac{L_f(1/2, \Pi^\sigma \times \Sigma^\sigma)}{p^\epsilon(\Pi^\sigma)p^\eta(\Sigma^\sigma)p_\infty(\mu,\lambda)}$$

from which the theorem follows. □

3.4. **The effect of changing rational structures.** In this section we study the effect of changing rational structures involved in the definition of the periods. Recall, from [33], that the period $p^{\epsilon}(\Pi)$ is defined by comparing a rational structure on the Whittaker model $W(\Pi_f)$ with that on a suitable cohomology space, namely, $H^{b_n}(\mathfrak{g}_\infty, K^0_\infty, V_\Pi \otimes M^\vee_\mu)(\epsilon)$. The rational structure on this cohomology space comes ultimately from a canonical $\mathbb{Z}$-structure on singular cohomology, however, the rational structure on the Whittaker model is not so canonical. In this section we draw attention to some other (very natural looking) rational structures on $W(\Pi_f)$. It should be borne in mind that a rational structure on $W(\Pi_f)$ is unique up to homotheties, so there is indeed an emphasis on the "naturality" of the definition.

For each $r \in \mathbb{Z}$, we define an action of $\mathrm{Aut}(\mathbb{C})$ on $W(\Pi_f)$ as follows: For $\sigma \in \mathrm{Aut}(\mathbb{C})$, $w \in W(\Pi_f, \psi)$, define

$$\sigma_r(w)(g) = \sigma\left(w\left(\begin{pmatrix} t_\sigma^{r-(n-1)} & & & \\ & t_\sigma^{r-(n-2)} & & \\ & & \cdots & \\ & & & t_\sigma^{r} \end{pmatrix} g\right)\right)$$

for all $g \in G_n(\mathbb{A}_f)$. It is easy to see that $w \mapsto \sigma_r(w)$ is a $G_n(\mathbb{A}_f)$-equivariant, σ-linear isomorphism from $W(\Pi, \psi)$ onto $W(\Pi^\sigma, \psi)$. For $r = 0$ this is nothing but the previous action we had considered. We can relate the two actions by pulling out a central character:

$$\sigma_r(w) = \sigma(\omega_\Pi(t_\sigma^r))\sigma_0(w) = \left(\frac{\sigma(\mathcal{G}(\omega_{\Pi_f}))}{\mathcal{G}(\omega_{\Pi_f^\sigma})}\right)^r \sigma_0(w). \tag{3.22}$$

If $w_0 \in W(\Pi)$ is the normalized new vector that is fixed by σ_0, for all $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q}(\Pi))$ then the vector

$$w_r := \mathcal{G}(\omega_{\Pi_f})^{-r} w_0$$

is fixed by all such σ_r. Hence the $\mathbb{Q}(\Pi)$-span of the $G_n(\mathbb{A}_f)$-orbit of w_r is the rational structure for this new action; we denote this rational structure by $W(\Pi_f)_r$. We have

$$W(\Pi_f)_r = \mathcal{G}(\omega_{\Pi_f})^{-r}\, W(\Pi_f)_0.$$

The comparison map $\mathcal{F}_{\Pi_f,\epsilon,[\Pi]_\infty} : W(\Pi_f) \to H(\Pi)$ is the same map as before. (For brevity, we abbreviate $H^{b_n}(\mathfrak{g}_\infty, K^0_\infty, V_\Pi \otimes M^\vee_\mu)(\epsilon)$ as $H(\Pi)$.) The normalization of this map is different, and we define a period $p^\epsilon_r(\Pi)$ by the requirement that the normalized map

$$\mathcal{F}^r = p^\epsilon_r(\Pi)^{-1}\mathcal{F}$$

maps the rational structure $W(\Pi)_r$ into the rational structure $H(\Pi)_0$; the latter being as before. As in [33, Definition/Proposition 3.3] one can give this definition in an $\mathrm{Aut}(\mathbb{C})$-equivariant manner. For these periods $p^\epsilon_r(\Pi)$, the main theorem of [33] looks like:

$$\sigma\left(\frac{p_r^{\epsilon\cdot\epsilon_\xi}(\Pi_f \otimes \xi_f)}{\mathcal{G}(\xi_f)^{n(n-1)/2-nr}\, p^\epsilon_r(\Pi_f)}\right) = \left(\frac{p_r^{\epsilon^\sigma\cdot\epsilon_{\xi^\sigma}}(\Pi_f^\sigma \otimes \xi_f^\sigma)}{\mathcal{G}(\xi_f^\sigma)^{n(n-1)/2-nr}\, p^{\epsilon^\sigma}(\Pi_f^\sigma)}\right)$$

for any algebraic Hecke character ξ of $\mathbb{Q}$.

It is tempting to stop at this moment and observe that if n is even, and we put $r = (n-2)/2$, then the periods $p^{\epsilon}_{(n-2)/2}(\Pi)$ have the same behaviour, upon twisting by Dirichlet characters, as the motivic periods of Deligne; the latter being known by Blasius [3] or Panchishkin [31]. However, it is not clear at the moment if $p^{\epsilon}_{(n-2)/2}(\Pi)$ indeed captures the possibly transcendental part of a critical value of the standard L-function of Π. We return to this theme about twisting in Section 4. We also formulate Conjecture 5.15 describing a relation between the periods of the type $p^{\epsilon}_0(\Pi)$ and Deligne's motivic periods.

Using the action σ_r, and the corresponding periods $p^{\epsilon}_r(\Pi)$, the main identity of Theorem 3.12 looks like:

$$\frac{L_f(1/2, \Pi \times \Sigma)}{p^{\epsilon}_r(\Pi)\, p^{\eta}_r(\Sigma)\, p_{\infty}(\mu, \lambda)} \;=\; \frac{\prod_{v \in S_\Sigma} L_v(1/2, \Pi_v \times \Sigma_v)}{\mathrm{vol}(\Sigma) \prod_{v \notin S_\Sigma} c_{\Pi_v}} \, \langle \vartheta^r_{\Sigma,\eta}, \vartheta^r_{\Pi,\epsilon} \rangle_{\mathcal{C}(R_f)},$$

with the classes defined as

$$\vartheta^r_{\Pi,\epsilon} = \mathcal{F}^r_{\Pi_f,\epsilon,[\Pi_\infty]}(w_{\Pi_f}), \quad \text{and} \quad \vartheta^r_{\Sigma,\eta} = \mathcal{F}^r_{\Sigma_f,\eta,[\Sigma_\infty]}(w_{\Sigma_f}),$$

where the global vectors w_{Π_f} and w_{Σ_f} are the same vectors as in 3.1.4. The action of σ on these classes can be read off using (3.22) and Proposition 3.15. In terms of the periods for σ_r, Theorem 1.1 on the central critical value now looks like:

$$\sigma\left(\frac{L_f(1/2, \Pi \times \Sigma)}{p^{\epsilon}_r(\Pi)\, p^{\eta}_r(\Sigma)\, \mathcal{G}(\omega_{\Pi_f})^r\, \mathcal{G}(\omega_{\Sigma_f})^{r+1}\, p_{\infty}(\mu, \lambda)}\right) = \frac{L_f(1/2, \Pi^\sigma \times \Sigma^\sigma)}{p^{\epsilon}_r(\Pi^\sigma)\, p^{\eta}_r(\Sigma^\sigma)\, \mathcal{G}(\omega_{\Pi^\sigma_f})^r\, \mathcal{G}(\omega_{\Sigma^\sigma_f})^{r+1}\, p_{\infty}(\mu, \lambda)}.$$

The moral of this section is an obvious one that one might have some freedom in defining periods, and proving relations amongst such periods, however, the L-functions are far more rigid; in the sense that the relations between L-values are more rigid than period relations.

4. Twisted L-functions

Given a cuspidal representation Π of $G_n(\mathbb{A})$, and a Dirichlet character χ, it is often of interest to know the behaviour of the critical values of $L(s, \Pi \otimes \chi)$ when we fix the critical point and the representation Π and let the character χ vary. One application of such a question is toward p-adic L-functions.

4.1. **A conjecture of Blasius and Panchishkin.** We now briefly review a conjecture independently due to Blasius [3, Conjecture L.9.8] and Panchishkin [31, Conjecture 2.3] about twisted L-values. Let Π be a regular algebraic cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$. We define $\eta(\Pi) \in \{\pm 1\}$ by

$$\eta(\Pi) = \mathrm{Tr}(\tau(\Pi_\infty)(j)),$$

where $\tau(\Pi_\infty)$ is the Langlands parameter of the representation Π_∞, which, we recall, is an n-dimensional semisimple representation of the Weil group $W_{\mathbb{R}} = \mathbb{C}^* \cup j\mathbb{C}^*$ of $\mathbb{R}$. Define $d^{\pm}(\Pi) \in \mathbb{Z}$ by

$$d^{\pm}(\Pi) = \begin{cases} n/2 & \text{if } n \text{ is even}, \\ (n \pm \eta(\Pi))/2 & \text{if } n \text{ is odd}. \end{cases}$$

Conjecture 4.1. *Let Π be a regular algebraic cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$. Let χ be an even Dirichlet character, which is thought of as a Hecke character. Note that both $L_f(s,\Pi)$ and $L_f(s,\Pi\otimes\chi)$ have the same set of critical points. Let m be such a common critical point; $m\in(n-1)/2+\mathbb{Z}$. We have*

$$L_f(m,\Pi\otimes\chi)\sim_{\mathbb{Q}(\Pi,\chi)}\mathcal{G}(\chi_f)^{d^{\pm}(\Pi\otimes\|\ \|^{(n-1)/2})}L_f(m,\Pi),$$

where $\mathbb{Q}(\Pi,\chi)$ denotes the number field generated by the values of the Dirichlet character χ and $\mathbb{Q}(\Pi)$; the sign $\pm$ is $(-1)^{m-(n-1)/2}$; and $d^{\pm}(\Pi\otimes\|\ \|^{(n-1)/2})$ is as defined above.

We note that if n is even, then the conjecture simplifies to

$$L_f(m,\Pi\otimes\chi)\sim_{\mathbb{Q}(\Pi,\chi)}\mathcal{G}(\chi_f)^{n/2}L_f(m,\Pi),$$

4.2. **Proof of Theorem 1.2.**

Proof. We now prove Theorem 1.2 about the behaviour of central critical value of Rankin–Selberg L-functions for $\mathrm{GL}_n\times\mathrm{GL}_{n-1}$ upon twisting by even Dirichlet characters. We go back to earlier notation. Note that Theorem 1.1 implies that

$$L_f(1/2,\Pi\times\Sigma)\ \sim_{\mathbb{Q}(\Pi,\Sigma)}\ p^{\epsilon}(\Pi)p^{\eta}(\Sigma)\mathcal{G}(\omega_{\Sigma_f})p_{\infty}(\mu,\lambda)$$

Let ξ be an even Dirichlet character, then the pair $(\Pi\otimes\xi,\Sigma)$ also satisfy the hypotheses of Theorem 1.1, with the same pair of highest weights (μ,λ), since ξ_∞ is trivial. Hence,

$$L_f(1/2,(\Pi\otimes\xi)\times\Sigma)\ \sim_{\mathbb{Q}(\Pi,\Sigma,\xi)}\ p^{\epsilon}(\Pi\otimes\xi)p^{\eta}(\Sigma)\mathcal{G}(\omega_{\Sigma_f})p_{\infty}(\mu,\lambda).$$

We invoke [33, Theorem 4.1] as rewritten in (2.2) to get

$$p^{\epsilon}(\Pi\otimes\xi)\sim_{\mathbb{Q}(\Pi,\xi)}\mathcal{G}(\xi_f)^{n(n-1)/2}p^{\epsilon}(\Pi).$$

Putting the above together gives

$$L_f(1/2,(\Pi\otimes\xi)\times\Sigma)\ \sim_{\mathbb{Q}(\Pi,\Sigma,\xi)}\ \mathcal{G}(\xi_f)^{n(n-1)/2}L_f(1/2,\Pi\times\Sigma).$$

□

4.3. **Some remarks.**

4.3.1. Note that in the proof of Theorem 1.2, we could have absorbed the twisting character ξ into Σ since

$$L_f(s,(\Pi\otimes\xi)\times\Sigma)=L_f(s,\Pi\times(\Sigma\otimes\xi)).$$

If we started with twisting Σ by ξ, then we would only get $\mathcal{G}(\xi_f)^{(n-1)(n-2)/2}$ by applying (2.2) to $p^{\eta}(\Sigma\otimes\xi)$. However, there is also the term involving the Gauss sum of ω_Σ, and since the central character transforms as $\omega_{\Sigma\otimes\xi}=\xi^{n-1}\omega_\Sigma$, from [37, Lemma 8] we have

$$\mathcal{G}(\xi_f^{n-1}\omega_{\Sigma_f})\sim_{\mathbb{Q}(\omega_\Sigma,\xi)}\mathcal{G}(\xi_f)^{n-1}\mathcal{G}(\omega_{\Sigma_f}),$$

i.e., we get the same net contribution of the Gauss sum.

4.3.2. *The Blasius-Panchishkin conjecture for some cusp forms on* GL_6. We record that Theorem 1.2 implies Conjecture 4.1 for certain cuspidal automorphic representations of $\mathrm{GL}_6(\mathbb{A})$.

Corollary 4.2. *Let* Π *(resp.,* Σ*) be a regular algebraic representation of* $\mathrm{GL}_3(\mathbb{A})$ *(resp.,* $\mathrm{GL}_2(\mathbb{A})$*). Let* $\Xi = \Pi \boxtimes \Sigma$ *be the automorphic representation of* $\mathrm{GL}_6(\mathbb{A})$ *which is the Kim-Shahidi transfer (for the* L*-homomorphism* $\mathrm{GL}_2 \times \mathrm{GL}_3 \to \mathrm{GL}_6$ *given by tensor product) of the pair* (Π, Σ)*. Assume that* Ξ *is regular and cuspidal (it is necessarily algebraic), and that* $s = 1/2$ *is critical for* $L_f(s, \Xi)$*. Then for any any even Dirichlet character* ξ *we have*

$$L_f(1/2, \Xi \otimes \xi) \ \sim_{\mathbb{Q}(\Xi,\xi)} \ \mathcal{G}(\xi_f)^3 L_f(1/2, \Xi).$$

Proof. The standard L-function $L(s, \Xi \otimes \xi)$ is nothing but the Rankin–Selberg L-function $L(s, \Pi \otimes \xi \times \Sigma)$. We leave the rest of the details to the reader. $\square$

Note that since Kasten and Schmidt [22] have recently proved Hypothesis 3.10 in the situation of $\mathrm{GL}_3 \times \mathrm{GL}_2$, the above *corollary is therefore true unconditionally.* We also note that using the cuspidality criterion of Ramakrishnan-Wang [35], and by taking Π and Σ to be regular with parameters unrelated to each other, one can see that the set of cuspidal representations Ξ to which the corollary applies is a nonempty set! We mention in passing that Qingyu Wu [41] has recently studied the image of this transfer.

5. Odd symmetric power L-functions

Let φ be a primitive holomorphic cusp form on the upper half plane of weight k, for $\Gamma_0(N)$, with nebentypus character ω. We denote this as $\varphi \in S_k(N, \omega)_{\mathrm{prim}}$. For any integer $r \geq 1$, consider the r-th symmetric power L-function $L_f(s, \mathrm{Sym}^r\varphi, \xi)$ attached to φ, twisted by a Dirichlet character ξ. In this section we prove Theorem 1.3 which gives an algebraicity theorem for certain critical values of such L-functions when r is an odd integer ≤ 7.

5.1. **Some preliminaries.**

5.1.1. *Symmetric power* L*-functions.* We will work with the L-function $L_f(s, \mathrm{Sym}^r\varphi, \xi)$ in the automorphic context, toward which we let $\pi(\varphi)$ be the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ attached to φ. For any integer $r \geq 1$, Langlands' functoriality predicts the existence of an isobaric automorphic representation $\mathrm{Sym}^r(\pi(\varphi))$ of $\mathrm{GL}_{r+1}(\mathbb{A})$, which is known to exist for $r \leq 4$ by the work of Gelbart and Jacquet [12], Kim and Shahidi [25], and Kim [24]. If $L(s, \mathrm{Sym}^r(\pi(\varphi)))$ denotes the standard L-function of $\mathrm{Sym}^r(\pi(\varphi))$, then we have

$$L_f(s, \mathrm{Sym}^r\varphi, \xi) = L_f(s - r(k-1)/2, \mathrm{Sym}^r(\pi(\varphi)) \otimes \xi).$$

For $r \geq 5$, Langlands' functoriality is not known for the r-th symmetric power, however, by the work of Kim and Shahidi [26], for $5 \leq r \leq 9$ one does have results about the analytic properties of these L-functions. Let S be any finite set of places containing archimedean and all ramified places for $\pi(\varphi)$, and define the partial L-functions $L^S(s, \pi(\varphi), \mathrm{Sym}^r \otimes \xi)$ as in [26, §4]. From [26, Proposition 4.2] and [26, Proposition 4.5] we have

(1) $L^S(s, \pi(\varphi), \mathrm{Sym}^5 \otimes \xi)$ is holomorphic and nonzero in $\mathrm{Re}(s) \geq 1$;
(2) $L^S(s, \pi(\varphi), \mathrm{Sym}^7 \otimes \xi)$ is holomorphic and nonzero in $\mathrm{Re}(s) \geq 1$.

For $v \in S$, one defines the local factors $L(s, \pi(\varphi)_v, \mathrm{Sym}^r \otimes \xi_v)$ via the local Langlands correspondence. After completing the partial L-functions with these local factors, one gets that both $L_f(s, \pi(\varphi), \mathrm{Sym}^5 \otimes \xi)$ and $L_f(s, \pi(\varphi), \mathrm{Sym}^7 \otimes \xi)$ are holomorphic and nonzero in $\mathrm{Re}(s) \geq 1$. By abuse of notation, we write

$$L_f(s, \pi(\varphi), \mathrm{Sym}^5 \otimes \xi) = L_f(s, \mathrm{Sym}^5(\pi(\varphi)) \otimes \xi),$$

and so also for the seventh symmetric power.

5.1.2. *Decomposition of certain Rankin-Selberg L-functions.*

Lemma 5.1. *Let σ be a two dimensional representation of some group. Then for $n \geq 2$*

$$\mathrm{Sym}^n(\sigma) \otimes \mathrm{Sym}^{n-1}(\sigma) \simeq \mathrm{Sym}^{2n-1}(\sigma) \oplus (\mathrm{Sym}^{2n-3}(\sigma) \otimes \det(\sigma)) \oplus \cdots \oplus (\sigma \otimes \det(\sigma)^{n-1})$$

Proof. This is Clebsch–Gordon for finite-dimensional representations of $\mathrm{GL}_2(\mathbb{C})$. □

Corollary 5.2. *Let $\varphi \in S_k(N, \omega)_{\mathrm{prim}}$, and let $\pi(\varphi)$ be the associated cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. Let $n \leq 4$. For $\mathrm{Re}(s) \geq 1$ we have*

$$L_f(s, \mathrm{Sym}^n(\pi(\varphi)) \times \mathrm{Sym}^{n-1}(\pi(\varphi))) = \prod_{a=1}^{n} L_f(s, \mathrm{Sym}^{2a-1}(\pi(\varphi)) \otimes \omega_{\pi(\varphi)}^{n-a}).$$

Assuming Langlands' functoriality, the above equality holds for all $n \geq 1$.

5.1.3. *Symmetric power transfers have nontrivial cohomology.* To apply Theorem 1.1 to get information about critical values for symmetric power L-functions, we need to know that the representation $\mathrm{Sym}^n(\pi(\varphi))$ has nontrivial cohomology. The following theorem is essentially due to Labesse and Schwermer [27]. (See also [32, §5].)

Theorem 5.3. *Let $\varphi \in S_k(N, \omega)_{\mathrm{prim}}$ with $k \geq 2$. Let $n \geq 1$. Assume that $\mathrm{Sym}^n(\pi(\varphi))$ is a cuspidal representation of $\mathrm{GL}_{n+1}(\mathbb{A})$. Let*

$$\Pi = \mathrm{Sym}^n(\pi(\varphi)) \otimes \xi \otimes \| \ \|^s,$$

where ξ is a Hecke character such that $\xi_\infty = \mathrm{sgn}^\epsilon$, with $\epsilon \in \{0, 1\}$, and $\| \ \|$ is the adèlic norm. We suppose that s and ϵ satisfy:

1. *If n is even, then let $s \in \mathbb{Z}$ and $\epsilon \equiv n(k-1)/2 \pmod 2$.*
2. *If n is odd then, we let $s \in \mathbb{Z}$ if k is even, and we let $s \in 1/2 + \mathbb{Z}$ if k is odd. We impose no condition on ϵ.*

Then $\Pi \in \mathrm{Coh}(G_{n+1}, \mu^\vee)$, where $\mu \in X_0^+(T_{n+1})$ is given by

$$\mu = \left(\frac{n(k-2)}{2} + s, \frac{(n-2)(k-2)}{2} + s, \ldots, \frac{-n(k-2)}{2} + s \right) = (k-2)\rho_{n+1} + s,$$

with ρ_{n+1} being half the sum of positive roots of GL_{n+1}.

5.2. **Proof of Theorem 1.3.** As mentioned above, the proof of Theorem 1.3 is obtained by applying Theorem 1.1 when Π and Σ are two consecutive symmetric power transfers of the representation $\pi(\varphi)$. We have already commented that these representations, up to some minor twisting, are cohomological. We need to check that the other hypotheses of Theorem 1.1, which concern the highest weights μ and λ, also hold for these choices. In the following proposition we record the various choices to be made, which depend on the parities of n and k. We also record the critical set for the Rankin–Selberg L-function at hand, and note that $s = 1/2$ is critical in all the cases we consider. Lastly, we specify the signs ϵ and η given by the recipe in Theorem 3.12 for the specific choice of representations in each case.

Proposition 5.4. *Let $\varphi \in S_k(N, \omega)$ be a primitive cusp form, and $\pi(\varphi)$ the associated cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. Let θ be any quadratic odd Dirichlet character. Let ξ be any Dirichlet character. (We think of θ and ξ as Hecke characters.)*

(1) *k-even ≥ 4, and n-even.*
  - $\Pi = \mathrm{Sym}^n(\pi(\varphi)) \otimes \theta^{n/2}$; $\mu = (k-2)\rho_{n+1}$; $\epsilon = (-1)^{n(n+1)/2}$.
  - $\Sigma = \mathrm{Sym}^{n-1}(\pi(\varphi)) \otimes \xi \otimes \| \ \|$; $\lambda = (k-2)\rho_n + 1$; $\eta = -\epsilon$.
  - *Critical set for* $L_f(s, \Pi \times \Sigma) = \{\frac{1-k}{2}, \frac{3-k}{2}, \dots, \frac{1}{2}, \dots, \frac{k-3}{2}\}$.

(2) *k-even ≥ 4 and n-odd.*
  - $\Pi = \mathrm{Sym}^n(\pi(\varphi)) \otimes \xi \otimes \| \ \|$; $\mu = (k-2)\rho_{n+1} + 1$; $\epsilon = \eta$.
  - $\Sigma = \mathrm{Sym}^{n-1}(\pi(\varphi)) \otimes \theta^{(n-1)/2}$; $\lambda = (k-2)\rho_n$; $\eta = (-1)^{n(n-1)/2}$.
  - *Critical set for* $L_f(s, \Pi \times \Sigma) = \{\frac{1-k}{2}, \frac{3-k}{2}, \dots, \frac{1}{2}, \dots, \frac{k-3}{2}\}$.

(3) *k-odd ≥ 3 and n-even.*
  - $\Pi = \mathrm{Sym}^n(\pi(\varphi))$; $\mu = (k-2)\rho_{n+1}$; $\epsilon = (-1)^{n(n+1)/2}$.
  - $\Sigma = \mathrm{Sym}^{n-1}(\pi(\varphi)) \otimes \xi \otimes \| \ \|^{1/2}$; $\lambda = (k-2)\rho_n + 1/2$; $\eta = -\epsilon$.
  - *Critical set for* $L_f(s, \Pi \times \Sigma) = \{\frac{2-k}{2}, \frac{4-k}{2}, \dots, \frac{1}{2}, \dots, \frac{k-2}{2}\}$.

(4) *k-odd ≥ 3 and n-odd.*
  - $\Pi = \mathrm{Sym}^n(\pi(\varphi)) \otimes \xi \otimes \| \ \|^{1/2}$; $\mu = (k-2)\rho_{n+1} + 1/2$; $\epsilon = \eta$.
  - $\Sigma = \mathrm{Sym}^{n-1}(\pi(\varphi))$; $\lambda = (k-2)\rho_n$; $\eta = (-1)^{n(n-1)/2}$.
  - *Critical set for* $L_f(s, \Pi \times \Sigma) = \{\frac{2-k}{2}, \frac{4-k}{2}, \dots, \frac{1}{2}, \dots, \frac{k-2}{2}\}$.

We add some comments to illuminate the various twistings and the assumptions on the weight k in the above proposition.

(1) Twisting by ξ. To apply Corollary 5.2 to get critical values of a certain odd symmetric power, one needs to know the critical values of smaller odd symmetric power L-functions *twisted* by certain characters.
(2) Twisting by a power of θ. The presence of this odd Dirichlet character is dictated by the vagaries of Theorem 5.3 in the case when both k and n are even.
(3) Twisting by $\| \ \|$ when k is even. This is an artifice introduced so that we are really working with the critical point $s = 3/2$ where all the L-functions at hand are nonvanishing. We need nonvanishing because to apply Corollary 5.2 we need to invert all but one of the factors on the right hand side. We could avoid this twist if we had a theorem about simultaneous nonvanishing of twisted L-functions at $s = 1/2$. As of now, the best available theorem along these lines seems to be due to Chinta-Friedberg-Hoffstein [5], but this is not able to handle the point $s = 1/2$.

(4) Twisting by $\| \ \|^{1/2}$ when k is odd. This is simply to ensure that we are working with an algebraic representation. This twist automatically takes care that we are dealing with L-functions at $s = 1$ where they are nonvanishing [18].
(5) If k is even (resp., odd) then we take $k \geq 4$ (resp., $k \geq 3$) so that the condition $\mu^\vee \succ \lambda$ is satisfied. In particular, that we do not say anything about the critical values of odd symmetric power L-functions of elliptic curves. We note that for $k = 1$, none of the symmetric power L-function have critical points! (See [32].)

Proof of Proposition 5.4. In each case, one has $\Pi \in \mathrm{Coh}(G_{n+1}, \mu^\vee)$ and $\Sigma \in \mathrm{Coh}(G_n, \lambda^\vee)$ by Theorem 5.3. The signs ϵ and η are given by Theorem 3.12. The list of critical points is an easy exercise involving the L-factors at infinity: one can write down the Langlands parameters of the representations Π_∞ and Σ_∞, and then write down $L(s, \Pi_\infty \times \Sigma_\infty)$. Now do the same with the dual representations, and the list follows in every case from the definition of a critical point. It should be kept in mind that $\Pi \times \Sigma$ is, via functoriality, a representation of $\mathrm{GL}_{n(n+1)}$, and $n(n+1)$ is even; the so-called motivic normalization dictates that one looks at critical points in $(n(n+1) - 1)/2 + \mathbb{Z} = 1/2 + \mathbb{Z}$. We omit the routine details. $\square$

For the proof of Theorem 1.3 we start with the case when k is even. Successively apply Theorem 1.1 for the pairs of representations $(\mathrm{Sym}^r(\pi(\varphi)), \mathrm{Sym}^{r-1}(\pi(\varphi)))$ for $r = 1, 2, 3, 4$, where the representations are taken with appropriate twisting characters as prescribed by Proposition 5.4. The proof repeatedly uses the period relations in (2.2), and the fact ([37, Lemma 8]) that for two Hecke characters χ_1 and χ_2 and $\sigma \in \mathrm{Aut}(\mathbb{C})$ one has $\sigma(\mathcal{G}(\chi_1\chi_2)/\mathcal{G}(\chi_1)\mathcal{G}(\chi_2)) = \mathcal{G}(\chi_1^\sigma\chi_2^\sigma)/\mathcal{G}(\chi_1^\sigma)\mathcal{G}(\chi_2^\sigma)$. We have the following:

$$L_f\left(\frac{3}{2}, \pi(\varphi) \otimes \xi\right) \sim p^{-\xi(-1)}(\pi(\varphi))\, \mathcal{G}(\xi)\, p_\infty((k-2)\rho_2 + 1). \tag{5.5}$$

$$L_f\left(\frac{3}{2}, \mathrm{Sym}^3(\pi(\varphi)) \otimes \xi\right) \sim p^+(\mathrm{Sym}^2(\pi(\varphi))) \frac{p^{\xi(-1)}(\pi(\varphi))}{p^{-\xi(-1)}(\pi(\varphi))}\, \mathcal{G}(\xi)^2\, \frac{p_\infty((k-2)\rho_3, (k-2)\rho_2 + 1)}{p_\infty((k-2)\rho_2 + 1)}. \tag{5.6}$$

$$L_f\left(\frac{3}{2}, \mathrm{Sym}^5(\pi(\varphi)) \otimes \xi\right) \sim \frac{p^{-\xi(-1)}(\mathrm{Sym}^3(\pi(\varphi)))}{p^{\xi(-1)}(\pi(\varphi))}\, \frac{\mathcal{G}(\xi)^3}{\mathcal{G}(\omega)}\, \frac{p_\infty((k-2)\rho_4 + 1, (k-2)\rho_3)}{p_\infty((k-2)\rho_3, (k-2)\rho_2 + 1)}. \tag{5.7}$$

$$L_f\left(\frac{3}{2}, \mathrm{Sym}^7(\pi(\varphi)) \otimes \xi\right) \sim \frac{p^+(\mathrm{Sym}^4(\pi(\varphi)))}{p^+(\mathrm{Sym}^2(\pi(\varphi)))}\, \frac{p^{\xi(-1)}(\mathrm{Sym}^3(\pi(\varphi)))}{p^{-\xi(-1)}(\mathrm{Sym}^3(\pi(\varphi)))}\, \frac{\mathcal{G}(\xi)^4}{\mathcal{G}(\omega)^3}\, \frac{p_\infty((k-2)\rho_5, (k-2)\rho_4 + 1)}{p_\infty((k-2)\rho_4 + 1, (k-2)\rho_3)}. \tag{5.8}$$

We omit the proof as it is an extended exercise in book-keeping. Similarly, when the weight k is odd, we get the following:

$$L_f\left(1, \pi(\varphi) \otimes \xi\right) \sim p^{\xi(-1)}(\pi(\varphi) \otimes \| \ \|^{1/2})\, \mathcal{G}(\xi)\, p_\infty((k-2)\rho_2 + 1/2). \tag{5.9}$$

(5.10)
$$L_f\left(1, \mathrm{Sym}^3(\pi(\varphi)) \otimes \xi\right) \sim p^-(\mathrm{Sym}^2(\pi(\varphi))) \frac{p^{\xi(-1)}(\pi(\varphi) \otimes \| \|^{1/2})}{p^{-\xi(-1)}(\pi(\varphi) \otimes \| \|^{1/2})} \mathcal{G}(\xi)^2 \frac{p_\infty((k-2)\rho_3, (k-2)\rho_2 + 1/2)}{p_\infty((k-2)\rho_2 + 1/2)}.$$

(5.11)
$$L_f\left(1, \mathrm{Sym}^5(\pi(\varphi)) \otimes \xi\right) \sim \frac{p^{-\xi(-1)}(\mathrm{Sym}^3(\pi(\varphi)) \otimes \| \|^{1/2})}{p^{-\xi(-1)}(\pi(\varphi) \otimes \| \|^{1/2})} \frac{\mathcal{G}(\xi)^3}{\mathcal{G}(\omega)} \frac{p_\infty((k-2)\rho_4 + 1/2, (k-2)\rho_3)}{p_\infty((k-2)\rho_3, (k-2)\rho_2 + 1/2)}.$$

(5.12)
$$L_f\left(1, \mathrm{Sym}^7(\pi(\varphi)) \otimes \xi\right) \sim \frac{p^+(\mathrm{Sym}^4(\pi(\varphi)))}{p^-(\mathrm{Sym}^2(\pi(\varphi)))} \frac{p^{-\xi(-1)}(\mathrm{Sym}^3(\pi(\varphi)) \otimes \| \|^{1/2})}{p^{\xi(-1)}(\mathrm{Sym}^3(\pi(\varphi)) \otimes \| \|^{1/2})} \frac{\mathcal{G}(\xi)^4}{\mathcal{G}(\omega)^3} \frac{p_\infty((k-2)\rho_5, (k-2)\rho_4 + 1/2)}{p_\infty((k-2)\rho_4 + 1/2, (k-2)\rho_3)}.$$

In all the above equations, by $\sim$ we mean up to algebraic quantities in an appropriate rationality field, namely the number field $\mathbb{Q}(\varphi, \xi)$. More generally, one can say that the quotient of the two sides is equivariant under $\mathrm{Aut}(\mathbb{C})$.

We note that the complex number $p^\epsilon(\varphi, 2n-1)$ in the statement of Theorem 1.3 is a combination of periods attached to various symmetric power representations. For example, from (5.8), one has

$$p^\epsilon(\varphi, 7) = \frac{p^+(\mathrm{Sym}^4(\pi(\varphi)))}{p^+(\mathrm{Sym}^2(\pi(\varphi)))} \frac{p^\epsilon(\mathrm{Sym}^3(\pi(\varphi)))}{p^{-\epsilon}(\mathrm{Sym}^3(\pi(\varphi)))} \mathcal{G}(\omega)^{-3}$$

when the weight of φ is even. By induction on n, it is possible to write down an expression for $p^\epsilon(\varphi, 2n-1)$. Similarly, one can write down an expression for $p(m,k)$ in terms of $p_\infty(\mu, \lambda)$ for various weights μ and λ. We omit the tedious details.

5.3. Twisted symmetric power L-functions.

5.3.1. *A special case of* [32, Conjecture 7.1]. In an earlier paper with Shahidi [32], we had formulated a conjecture about the behaviour of the special values of symmetric power L-functions upon twisting by Dirichlet characters. See [32, Conjecture 7.1]. We note that Theorem 1.3 implies this conjecture for certain odd symmetric power L-functions.

Corollary 5.13. *Let φ and the critical point m be as in Theorem 1.3. Let ξ be an even Dirichlet character which we identify with the corresponding Hecke character. For $n \leq 4$ we have*
$$L_f(m, \mathrm{Sym}^{2n-1}\varphi, \xi) \ \sim \ \mathcal{G}(\xi_f)^n L_f(m, \mathrm{Sym}^{2n-1}\varphi),$$
where, by $\sim$, we mean up to an element of the number field $\mathbb{Q}(\varphi, \xi)$. Moreover, the quotient $L_f(m, \mathrm{Sym}^{2n-1}\varphi, \xi)/(\mathcal{G}(\xi_f)^n L_f(m, \mathrm{Sym}^{2n-1}\varphi))$ is $\mathrm{Aut}(\mathbb{C})$-equivariant.

5.3.2. *Conjecture 4.1 plus Langlands' functoriality implies* [32, Conjecture 7.1]. Our conjecture on twisted symmetric power L-values follows from the more general conjecture of Blasius and Panchishkin. We note that the heuristics on the basis of which we formulated [32, Conjecture 7.1] are entirely disjoint from the motivic calculations of Blasius and Panchishkin which is the basis of Conjecture 4.1. In this subsection, we briefly sketch a proof of how Conjecture 4.1 plus Langlands' functoriality for the L-homomorphism $\mathrm{Sym}^n : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_{n+1}(\mathbb{C})$ implies [32, Conjecture 7.1].

Proposition 5.14. *Let $\varphi \in S_k(N,\omega)_{\mathrm{prim}}$. Let $n \geq 1$ be any integer, χ an even Dirichlet character (identified with a Hecke character), and m a critical integer for $L_f(s, \mathrm{Sym}^n\varphi, \chi)$. Then, assuming Langlands functoriality in as much as assuming that $\mathrm{Sym}^n(\pi(\varphi))$ exists as an automorphic representation of $\mathrm{GL}_{n+1}(\mathbb{A})$, Conjecture 4.1 implies*

$$L_f(m, \mathrm{Sym}^n\varphi, \chi) \sim \mathcal{G}(\chi_f)^{\lceil (n+1)/2 \rceil} L_f(m, \mathrm{Sym}^n\varphi),$$

unless n is even and m is odd (to the left of center of symmetry), in which case we have

$$L_f(m, \mathrm{Sym}^n\varphi, \chi) \sim \mathcal{G}(\chi_f)^{n/2} L_f(m, \mathrm{Sym}^n\varphi),$$

where $\sim$ means up to an element of $\mathbb{Q}(\varphi, \chi)$.

Proof. We assume that $\mathrm{Sym}^n(\pi(\varphi))$ is cuspidal, because, if not, then φ is either dihedral or a form of weight 1. (This follows from Kim–Shahidi [26] and Ramakrishnan [34].) If φ is dihedral, then we have verified the conclusion; indeed this was one of the heuristics for [33, Conjecture 7.1]. If it has weight 1, then none of the symmetric power L-functions have critical integers ([32, Remark 3.8]) and so the conclusion is vacuously true!

Let $\Pi = \mathrm{Sym}^n(\pi(\varphi))$. The restriction to $\mathbb{C}^*$ of the Langlands parameter of Π_∞ is given by

$$z \mapsto \bigoplus_{i=0}^{n} z^{(n-2i)(k-1)/2}\, \bar{z}^{(2i-n)(k-1)/2}.$$

Note that Π is algebraic if and only if $(n-2i)(k-1)/2 + n/2$ is an integer, and this is so if and only if nk is even. If both n and k are odd then $\Pi \otimes \| \, \|^{1/2}$ is algebraic.

We start with the case when n is even. Applying Conjecture 4.1 we have

$$\begin{aligned} L_f(m, \mathrm{Sym}^n\varphi, \chi) &= L_f(m - n(k-1)/2, \mathrm{Sym}^n(\pi(\varphi)) \otimes \chi), \\ &\sim \mathcal{G}(\chi_f)^{\frac{n+1 \pm \eta(\Pi \otimes \| \, \|^{n/2})}{2}} L_f(m - n(k-1)/2, \mathrm{Sym}^n(\pi(\varphi))) \\ &\sim \mathcal{G}(\chi_f)^{\frac{n+1 \pm \eta(\Pi \otimes \| \, \|^{n/2})}{2}} L_f(m, \mathrm{Sym}^n\varphi), \end{aligned}$$

where $\pm = m - n(k-1)/2 - n/2 = m - nk/2$. Now we compute $\eta(\Pi \otimes \| \, \|^{n/2})$ toward which one can check that the Langlands parameter of Π_∞ is given by

$$\mathrm{Sym}^n(I(\chi_{k-1})) = \epsilon^{n(k-1)/2} \oplus \bigoplus_{a=1}^{n/2} I(\chi_{(2a(k-1))}),$$

where for any integer b, $I(\chi_b)$ denotes the induction to $W_{\mathbb{R}}$ of the character $z \mapsto (z/|z|)^b$ of $\mathbb{C}^*$, and ϵ is the sign character of $\mathbb{R}^*$ which is thought of as a character of $W_{\mathbb{R}}$ via the isomorphism $W_{\mathbb{R}}^{\mathrm{ab}} \simeq \mathbb{R}^*$. It is easy to see that on all the two dimensional summands the element $j \in W_{\mathbb{R}}$ has trace equal to 0, and ϵ maps j to -1. We get

$$\eta(\Pi \otimes \| \, \|^{n/2}) = (-1)^{nk/2}.$$

From this we get

$$L_f(m, \mathrm{Sym}^n\varphi, \chi) \sim \mathcal{G}(\chi_f)^{\frac{n+1 \pm (-1)^{nk/2}}{2}} L_f(m, \mathrm{Sym}^n\varphi).$$

We contend that from here on it is easy to see that the conclusion follows. (It might help the reader to further subdivide into the cases depending on when k is even or odd.)

If n is odd then the exponent of the Gauss sum that factors out is predicted to be $(n+1)/2$ $(= d^{\pm}(\Pi))$ in both Conjecture 4.1 and so also in the conclusion of the proposition. One detail that needs to be circumvented is that Π is not algebraic if both n and k are odd; for this case we argue as:

$$\begin{array}{rcl} L_f(m, \mathrm{Sym}^n\varphi, \chi) & = & L_f(m - n(k-1)/2 - 1/2, \mathrm{Sym}^n(\pi(\varphi)) \otimes \| \ \|^{1/2} \otimes \chi) \\ & \sim & \mathcal{G}(\chi_f)^{\frac{n+1}{2}} L_f(m - n(k-1)/2 - 1/2, \mathrm{Sym}^n(\pi(\varphi)) \otimes \| \ \|^{1/2}) \\ & = & \mathcal{G}(\chi_f)^{\frac{n+1}{2}} L_f(m, \mathrm{Sym}^n\varphi). \end{array}$$

$\square$

5.4. **Remarks on compatibility with Deligne's conjecture.** We recall the famous conjecture of Deligne [9, Conjecture 2.8]. Let M be a motive, and assume that $s = 0$ is critical for the L-function $L(s, M)$. Deligne attaches two periods $c^{\pm}(M)$ to M by comparing the Betti and de Rham realizations of M. He predicts that $L(0, M)/c^{+}(M)$ is in a suitable number field $\mathbb{Q}(M)$, and more generally, the ratio is $\mathrm{Aut}(\mathbb{C})$-equivariant. One expects that our theorems above are compatible with Deligne's conjecture. This expectation is formalized in the following conjecture.

Conjecture 5.15 (Period relations)**.** *Let Π and Σ be regular algebraic cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ and $\mathrm{GL}_{n-1}(\mathbb{A})$, respectively. Let μ, λ be the associated highest weights, and ϵ, η the associated signs as in Theorem 1.1. We let $M(\Pi)$ and $M(\Sigma)$ be the conjectural motives attached to Π and Σ (by Clozel* [6, Conjecture 4.5]*). Let $M = M(\Pi) \otimes M(\Sigma)$. Let $c^{\pm}(M)$ be Deligne's periods attached to M, and $d^{\pm}(M)$ be the integers as in Deligne* [9, §1.7]*. We expect*

$$p^{\epsilon}(\Pi)p^{\eta}(\Sigma)\mathcal{G}(\omega_{\Sigma_f})p_{\infty}(\mu, \lambda) \sim (2\pi i)^{d^+(M)n(n-1)/2} c^{(-1)^{n(n-1)/2}}(M),$$

where, by $\sim$, we mean up to an element of the number field $\mathbb{Q}(\Pi, \Sigma)$.

The heuristic for the above conjecture is the following 'formal' calculation based on Theorem 1.1, Langlands' functoriality, the correspondence between automorphic representations and motives as in Clozel [6, Conjecture 4.5], and Deligne [9, Conjecture 2.8]:

$$\begin{array}{rcl} p^{\epsilon}(\Pi)p^{\eta}(\Sigma)\mathcal{G}(\omega_{\Sigma})p_{\infty}(\mu, \lambda) & \sim & L(1/2, \Pi \times \Sigma), \quad \text{(by Theorem 1.1)} \\ & = & L(1/2, \Pi \boxtimes \Sigma), \quad \text{(by Langlands' functoriality)} \\ & = & L(n(n-1)/2, M(\Pi \boxtimes \Sigma)), \quad \text{(motivic normalization)} \\ & = & L(n(n-1)/2, M), \quad \text{(by definition of } M) \\ & = & L(0, M(n(n-1)/2)), \quad \text{(see [9, 3.1.2])} \\ & \sim & c^{+}(M(n(n-1)/2)), \quad \text{(by Deligne [9, Conjecture 2.8])} \\ & = & (2\pi i)^{d^+(M)n(n-1)/2} c^{(-1)^{n(n-1)/2}}(M) \quad \text{(see [9, 5.1.8])}. \end{array}$$

It is possible to express $c^{\pm}(M(\Pi) \otimes M(\Sigma))$ in terms of the periods, or perhaps some other finer invariants, attached to $M(\Pi)$ and $M(\Sigma)$, as in Blasius [2] and Yoshida [42].

References

[1] A. Ash and D. Ginzburg, *p-adic L-functions for* $\mathrm{GL}(2n)$. Invent. Math. 116, no. 1-3, 27–73 (1994).

[2] D. Blasius, Appendix to Orloff Critical values of certain tensor product L-functions. Invent. Math. 90, no. 1, 181–188 (1987).

[3] D. Blasius, *Period relations and critical values of L-functions.* Olga Taussky-Todd: in memoriam. Pacific J. Math., Special Issue, 53–83 (1997).

[4] A. Borel and H. Jacquet, *Automorphic forms and automorphic representations.* Proc.Sympos. Pure Math., XXXIII, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, pp. 189–207, Amer. Math. Soc., Providence, R.I., 1979.

[5] G. Chinta, S. Friedberg and J. Hoffstein, *Asymptotics for sums of twisted L-functions and applications.* In Automorphic representations, L-functions and applications: progress and prospects, 75–94, Ohio State Univ. Math. Res. Inst. Publ., 11, de Gruyter, Berlin, 2005.

[6] L. Clozel, *Motifs et formes automorphes: applications du principe de fonctorialité.* (French) [Motives and automorphic forms: applications of the functoriality principle] Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), 77–159, Perspect. Math., 10, Academic Press, Boston, MA, 1990.

[7] J. Cogdell, *Lectures on L-functions, converse theorems, and functoriality for* GL_n. Lectures on automorphic L-functions, 1–96, Fields Inst. Monogr., 20, Amer. Math. Soc., Providence, RI, 2004.

[8] J. Cogdell and I.I. Piatetski-Shapiro, *Remarks on Rankin-Selberg convolutions.* Contributions to automorphic forms, geometry, and number theory, 255–278, Johns Hopkins Univ. Press, Baltimore, MD, 2004.

[9] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales (French),* With an appendix by N. Koblitz and A. Ogus. Proc. Sympos. Pure Math., XXXIII, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, pp. 313–346, Amer. Math. Soc., Providence, R.I., 1979.

[10] W.T. Gan, B. Gross and D. Prasad, *Symplectic local root number, central critical L-values, and restriction problems in the representation theory of classical groups.* Preprint (2008).

[11] P. Garrett and M. Harris, *Special values of triple product L-functions.* American J. Math., Vol 115, 159–238 (1993).

[12] S. Gelbart and H. Jacquet, *A relation between automorphic representations of* $\mathrm{GL}(2)$ *and* $\mathrm{GL}(3)$. Ann. Sci. École Norm. Sup. (4) 11, no. 4, 471–542 (1978).

[13] R. Godement, *Notes on Jacquet–Langlands' Theory.* Preprint, I.A.S Princeton, (1970).

[14] P. Griffiths and J. Harris, *Principles of algebraic geometry.* Reprint of the 1978 original. Wiley Classics Library. John Wiley and Sons, Inc., New York, 1994. xiv+813 pp.

[15] G. Harder, *General aspects in the theory of modular symbols.* Seminar on number theory, Paris 1981–82 (Paris, 1981/1982), 73–88, Progr. Math., 38, Birkhuser Boston, Boston, MA, 1983.

[16] M. Harris, *Occult period invariants and critical values of the degree four L-function of* $\mathrm{GSp}(4)$. Contributions to automorphic forms, geometry, and number theory, 331–354, Johns Hopkins Univ. Press, Baltimore, MD, 2004.

[17] G. Henniart, *Sur la conjecture de Langlands locale pour* GL_n. 21st Journées Arithmétiques (Rome, 2001). J. Théor. Nombres Bordeaux 13, no. 1, 167–187 (2001).

[18] H. Jacquet and J. Shalika. *A non-vanishing theorem for zeta functions of* GL_n. Invent. Math. 38, no. 1, 1–16 (1976/77).

[19] H. Jacquet and J. Shalika. *On Euler products and the classification of automorphic representations I and II,* Amer. J. of Math., 103, 499-558 and 777–815 (1981).

[20] H. Jacquet, I. Piatetski-Shapiro and J. Shalika. *Conducteur des représentations du groupe linéaire.* (French) [Conductor of linear group representations] Math. Ann. 256, no. 2, 199–214 (1981).

[21] H. Jacquet, I.I. Piatetskii-Shapiro, J.A. Shalika, *Rankin–Selberg Convolutions.* American J. of Math., Vol. 105, No. 2, pp. 367–464 (1983).

[22] H. Kasten and C.-G. Schmidt, *On critical values of Rankin–Selberg convolutions.* Preprint (2008). Available at `http://www.mathematik.uni-karlsruhe.de/user/mathnet/preprint.html`

[23] D. Kazhdan, B. Mazur, C.-G. Schmidt, *Relative modular symbols and Rankin–Selberg convolutions.* J. Reine Angew. Math. 519, 97–141 (2000).

[24] H. Kim, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2. With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak. J. Amer. Math. Soc. 16, no. 1, 139–183 (2003).

[25] H. Kim and F. Shahidi, *Functorial products for* $\mathrm{GL}_2 \times \mathrm{GL}_3$ *and the symmetric cube for* GL_2*. With an appendix by Colin J. Bushnell and Guy Henniart.* Ann. of Math. (2) 155, no. 3, 837–893 (2002).

[26] H. Kim and F. Shahidi, *Cuspidality of symmetric powers with applications.* Duke Math. J. 112, no. 1, 177–197 (2002).

[27] J.-P. Labesse and J. Schwermer, *On liftings and cusp cohomology of arithmetic groups.* Invent. Math., 83, 383–401 (1986).

[28] J. Mahnkopf, *Modular symbols and values of L-functions on* GL_3. J. Reine Angew. Math. 497, 91–112 (1998).

[29] J. Mahnkopf, *Cohomology of arithmetic groups, parabolic subgroups and the special values of L-functions of* GL_n. J. Inst. Math. Jussieu. 4, no. 4, 553-637 (2005).

[30] J. Neukirch, *Algebraic number theory,* Die Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 322. Springer-Verlag, Berlin, (1999).

[31] A.A. Panchishkin, *Motives over totally real fields and p-adic L-functions.* Ann. Inst. Fourier (Grenoble) 44, no. 4, 989–1023 (1994).

[32] A. Raghuram and F. Shahidi, *Functoriality and special values of L-functions,* Eisenstein series and Applications, Progress in Mathematics 258. Boston, MA: Birkhäuser Boston, 2008.

[33] A. Raghuram and F. Shahidi, *On certain period relations for cusp forms on* GL_n, Int. Math. Res. Notices, (2008) Vol. 2008, article ID rnn077, 23 pages, doi:10.1093/imrn/rnn077.

[34] D. Ramakrishnan, *Remarks on the symmetric powers of cusp forms on* GL_2, Preprint, (2007), Available at `http://xxx.lanl.gov/abs/0710.0676`.

[35] D. Ramakrishnan and S. Wang, *A cuspidality criterion for the functorial product on* $\mathrm{GL}(2) \times \mathrm{GL}(3)$ *with a cohomological application.* Int. Math. Res. Not., no. 27, 1355–1394 (2004).

[36] C.-G. Schmidt, *Relative modular symbols and p-adic Rankin–Selberg convolutions,* Invent.Math. 112, 31–76 (1993).

[37] G. Shimura, *The special values of the zeta functions associated with cusp forms,* Comm. Pure Appl. Math., 29, no. 6, 783–804, (1976).

[38] G. Shimura, *On the periods of modular forms,* Math. Ann., 229, no. 3, 211–221, (1977).

[39] J. Tate, *Fourier analysis in number fields and Hecke's zeta function.* 1967 Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965) pp. 305–347 Thompson, Washington, D.C.

[40] A. Weil, *Basic number theory,* Die Grundlehren der mathematischen Wissenschaften, Band 144 Springer-Verlag New York, Inc., New York (1967).

[41] Qingyu Wu, *Image of transfer from* $\mathrm{GL}_2 \times \mathrm{GL}_3$ *to* GL_6. Ph.D. Thesis, Purdue University, (2008).

[42] H. Yoshida, *Motives and Siegel modular forms.* Amer. J. Math. 123, no. 6, 1171–1197 (2001).

Address:
A. Raghuram
Department of Mathematics
Oklahoma State University
401 Mathematical Sciences
Stillwater, OK 74078, USA.
E-mail address: `araghur@math.okstate.edu`